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SOUND PRODUCTION IN A MOVING STREAM

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CONTENTS

This work is a generalization of Lighthill's acoustic analogy in which it is proved that the jet noise problem can be modelled exactly by equivalent sources near a vortex sheet. Mani's work has shown that this type of scheme can agree well with experiment. This theory justifies Mani's general procedure but gives in addition the equivalent sources needed for an exact analogy. Each moving fluid particle supports a quadrupole whose strength per unit mass is given by Lighthill's stress tensor and the sound radiates as if it were adjacent to a laminar instability free vortex sheet. Though we show that the sound is determined in terms of the turbulence stress tensor, sound is also generated by the flow's instability waves as they grow into turbulence, and this sound appears as an exponentially growing precursor of the main field. Some well known features of the mean flow acoustic interaction issue are an immediate consequence of the theory. We examine the case of a round jet in some detail and concentrate on an aspect that we think is new. When the mean jet density is much lower than that of its environment then the mean flow–acoustic interaction results in a considerable amplification of the quadrupole field, and the intensity of its sound can scale on an unusually low power of jet speed. We show that a fourth power law is possible and even a second power law when the density difference is large enough. This may be part of the 'excess noise' problem in which the sound of engine-produced hot jets is often insensitive to changes in jet speed at low exhaust power.

1. Introduction

Since the experimental work of Lush (1971) there has been no doubt that the interaction of aerodynamic sound with the moving flow whose mixing provided its source is an important element of the jet noise problem. Many of the features described by Ribner (1964) as refraction of sound by gradients in mean flow and refractive index were confirmed in Lush's work, so putting experimental observation and intuitive reasoning considerably in the lead of any formal theory.

Of course it was always evident that some refraction of sound passing through an inhomogeneous flow was inevitable, and Lighthill (1952) argued that such effects should be treated separately after the sound *generation* problem had first been solved. Elementary jet refraction problems had been worked by Moretti & Slutsky (1959), Gottlieb (1960) and Slutsky & Tamagno (1961), each with evidence that flow could deform any sound wave passing through it. Only Lighthill's (1952) theory could treat the sound generation problem exactly, but that theory was incapable of explicitly displaying the refractive effects. Theories that do bring out the flow acoustic interaction explicitly are invariably approximate and difficult enough to handle for the consequences of the necessary approximations to remain obscure. For example, Gottlieb's work concerned a plane vortex sheet modelling of a jet, while Moretti & Slutsky described high frequency rays propagating through a laminar flow. Zones of silence arise in such flows and the upstream deflection of an initially downstream propagating ray is impossible. But some such upstream deflection is quite probable for high frequency waves in an actual jet as concentrated eddies can bend rays in a different and more extreme manner from that found in parallel laminar flows (cf. Dowling 1975; Howe 1976; Broadbent 1977).

Phillips (1960) was the first to attempt an exact description of sound generation by turbulent shear layers in which the flow interaction was made explicit. That work was extended by Pao (1969) and Lilley (1971). Primarily aimed at the supersonic problem, it left obscure the important issues that Mani (1976) emphasized; the Lighthill analogy did not seem to provide even an approximate low speed estimate of the full convection effects on the sound generated by sources shrouded in a mean jet flow. But Mani's analysis, the results of which seem to be so consistent with experiment in precisely those areas where Lush had pointed to deficiencies of the Lighthill theory, was based only on an intuitive modelling of the flow, some support for the model being deduced from a theory of Lilley's (1974). The obvious logical weakness in Mani's experimentally effective model led one to suspect that a more fundamental justification is possible; that is essentially what this paper is about.

A substantial amount of recent theoretical work on the interaction between the mean flow, the turbulence, and the acoustic field is based on the idea that the unsteady stream can be thought of as a weak perturbation to a parallel laminar inviscid flow with a velocity and density profile equal to that of the *mean* jet. That motion is of course governed by the compressible version of the Rayleigh equation, whose inhomogeneous form is the Lilley (1974) equation. It was this equation on which Mani argued the reasonableness of his own vortex sheet modelling, though in doing so he had to discard singular source terms that would actually dominate in any physical realization of that model. It was through this equation also that Mani (1976) argued the existence of efficient monopoles arising from mean density gradients in the jet, an argument that is now known to be incorrect (cf. Kempton 1976) and which was in fact an erroneous application of Lilley's equation (cf. Morfey & Tester 1976). Misapplications of the theory can be expected because the equation is not easy to handle; it is a third order differential equation with variable coefficients, solutions to which have to be sought numerically in all but the simplest of geometries.

There are difficulties of principle also. Non-trivial solutions of the homogeneous equation exist, indeed they are the ones so extensively studied in examining the stability of laminar flows, and because these do not depend on the 'right hand side' of Lilley's equation for their existence, that right hand side cannot claim to be the source; it is certainly not their source. Another difficulty of principle concerns the idea of evaluating the field as a weak perturbation about a laminar flow with the *mean* jet profile. Real tubulent jets are highly disturbed, the instantaneous velocity vector deviating from the axial direction by more than $\frac{1}{4}\pi$ quite frequently (Acton 1976). Fluctuations in a low Mach number jet are on the same time scale as the sound they produce, during which a sound wave travels many shear layer thicknesses. Any single sound wave therefore interacts with a geometrically contorted shear layer, quite unlike the mean shear layer, and the interaction is complete long before the shear layer has wobbled enough to resemble the mean. It might be thought that the mean effect of acoustic–shear layer interaction could be approximated to by a single passage of sound through a 'mean' shear layer. But that too seems unlikely, at least at high frequencies, because, for example, of the existence of shadow zones in the laminar shear layer problem that would be penetrated by rays in an actual case. In those cases, if a solution via the Lilley equation is correct, the Green functions of that equation cannot represent the real field very accurately and careful attention must be given to the detailed evaluation of the then necessarily extensive 'source terms'. The equation is difficult enough to handle and the instability issue so intractable that definite results are hard to obtain. Progress is not impossible though, as Morfey & Tester's (1976) work testifies. This difficulty of quantifying errors is even more pronounced in Mani's approach because he has no recipe for deducing what the equivalent sources should be.

Lilley's equation and the aero-acoustic analogy it provides is attracting great attention and is formally exact so that it would be wrong to concentrate on the difficulty of obtaining and interpreting the results. This laminar flow model is probably the best that is available, and it does account for the major effects of refraction remarkably well. The existence of quiet zones is an obvious element in which the qualitative similarity is good and away from the shadow zones there is quantitative agreement.

Now in many respects Mani's (1976) particular vortex sheet modelling of the jet problem seems as good as that more laboriously obtained via Lilley's equation (Morris 1974; Morfey & Tester 1976) in which the jet flow is given the mean velocity and temperature profile. Add to this the observation that the mean profile is no more relevant than any other since no single sound wave ever interacts with it and one is led to the thought that in those areas where the laminar flow theories are successful, details of the profile are actually unimportant. In that case, the best model will be that which is most easily handled. It is difficult to imagine any simpler model than a vortex sheet. It is this analytical tractability that might give Mani's model its edge; its weakness lies in its inability to define the sources. Because it is not based on an exact analogy it cannot provide a convincing procedure for handling the more subtle and difficult problems such as vortex sheet instability and the apparent singularities at the Mach wave conditions encountered at supersonic jet speeds.

In this paper we develop an exact analogy between a vortex sheet simulation of a jet flow and the real thing, and show that the equivalent source required in the analogy is a convected quadrupole distribution, each quadrupole having a strength determined by Lighthill's stress tensor measured relative to a uniformly moving jet stream. This result is essentially that already given by Ffowcs Williams (1974) but we now give a much more rigorous and convincing derivation of that theory.

Our vortex sheet extension of Lighthill's acoustic analogy contains elements that arrive at the far field before any wave could have covered the distance at the *uniform* propagation speeds of the model problem. But then the real wave is *not* travelling at these uniform speeds; it is riding on the back of turbulent eddies and refracting through regions of inhomogeneous refractive index. The exactly analogous model flow in which these real effects are suppressed must have a quasi-non-physical element through which they are restored. In this respect our extension is not different from Lighthill's model where wave elements travel at a strictly uniform speed. Sound travelling a unit length through a real turbulent flow of characteristic Mach number M, can be advanced, or retarded, a distance of order M by these convective effects, and similar arguments hold for variations in the speed of sound (cf. Ffowcs Williams 1977). Errors of order MD in the wave's position where D is the scale of the turbulent flow, are thus to be expected in all analogies of this type and ours is no exception. However, our exact analogy has an added feature brought about by a requirement that the linear model should share the real flow's finite level of acoustic activity. Any instability of the model flow must therefore be avoided, and this step also causes waves in the model to precede any that would exist if the linear model flow were actually to be realized in practice. We argue that this is entirely reasonable, for the sound producing disturbance would also provoke any real shear layer to support rapidly growing waves that later break into turbulence. Any theory that attributes the source of that sound to the turbulence must therefore allow for the sound's prior existence. In our model it is as if the source-induced vortex sheet instability is exactly the opposite of a pre-existing instability wave that is growing on the shear layer in anticipation of the wave it is eventually to cancel. The pre-existing wave has an exponential growth rate so that the anticipatory sound induced by that wave is significant only within a distance of the actual wave front that sound travels during the build up, a build up that persists over the characteristic time scale of the turbulence. This is also, necessarily, the characteristic time scale of the dominant flow instabilities which both feed the turbulence and determine the structure of the Green function for the model. Because in low Mach number flows this is also the characteristic time scale of the sound, the wave front is thus diffused forwards a distance of about one acoustic wavelength. The sound field therefore furnishes in this way a record of how the turbulence grew exponentially out of the instability waves; the turbulence provides the dominant source while the contribution made by instability waves growing into turbulence necessarily precedes it. The analogy, which we emphasize to be exact, allows both these elements to be specified through Lighthill's stress tensor and the bounded Green function for a vortex sheet simulation of the flow.

We first develop the theory in a general way and illustrate its application for a cylindrical jet of circular cross section and then derive from the theory some important known results for an acoustically compact jet flow. We contend that this theory provides a justification for the correctness of Mani's (1976) vortex sheet modelling. The sources needed for that model are however quite different from those used by Mani, which in any case are now known to be incorrect, at least in so far as they misrepresent the effects of mean density gradients. It is our belief that this theory allows all the power and simplicity of Lighthill's analogy to be applied to real jet flows in which there are significant mean flow acoustic interactions; these interactions are made explicit in an analytically tractable way.

We also give some new results for the acoustically compact jet of low density. There it seems that sound will scale on very low powers of jet speed. We think that this may well be part of the excess noise problem where sound is known not to conform with the simple scaling of Lighthill's free quadrupole model.

2. A generalization of Lighthill's acoustic analogy

Our development is based on Lighthill's (1952) equation

$$\frac{\partial^2 \rho'}{\partial \tau^2} - c_0^2 \frac{\partial^2 \rho'}{\partial y_i^2} = \frac{\partial^2 T_{ij}}{\partial y_i \, \partial y_j}, \tag{2.1}$$

where

$$T_{ij} = \rho v_i v_j + p_{ij} - c_0^2 \rho' \delta_{ij}; \quad p_{ij} = (p - p_0)\, \delta_{ij} - e_{ij} \tag{2.2}$$

is the turbulence stress tensor, v_i the fluid velocity, p the pressure, ρ the density, $\rho' = (\rho - \rho_0)$ and e_{ij} the viscous stress tensor. The suffix zero implies a constant reference value which can actually be chosen arbitrarily. It is well known that this equation is a direct consequence of the continuity and momentum equations

$$\left.\begin{aligned} \frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial y_i}(\rho v_i) &= 0, \\ \frac{\partial}{\partial \tau}(\rho v_i) + \frac{\partial}{\partial y_j}(T_{ij} + c_0^2 \rho' \delta_{ij}) &= 0. \end{aligned}\right\} \tag{2.3}$$

Lighthill's formulation of the aerodynamic sound problem is most appropriate when the sound generated by the flow propagates through a medium which is homogeneous and at rest relative to the observer. However, in many important situations it is often more realistic to regard the undisturbed medium or a particular part of that medium as being in uniform motion. For this we introduce a new coordinate system $\boldsymbol{y}'$ which moves in the y_1-direction with the uniform velocity $\boldsymbol{U} = (U_1, 0, 0)$.

$$y_i' = y_i - U_i \tau = y_i - U_1 \tau \delta_{i1}. \tag{2.4}$$

It follows from the Galilean invariance of the conservation principles embodied in equation (2.1) that

$$\left.\frac{\partial^2 \rho_1'}{\partial \tau^2}\right|_{y'} - c_1^2 \frac{\partial^2 \rho_1'}{\partial y_i'^2} = \frac{\partial^2 T_{ij}'}{\partial y_i' \, \partial y_j'} \tag{2.5}$$

where $\rho_1' = (\rho - \rho_1)$ and

$$T_{ij}' = \rho v_i' v_j' + p_{ij} - c_1^2 \rho_1' \delta_{ij} \tag{2.6}$$

is Lighthill's stress tensor expressed in terms of the relative velocity $v_i' = v_i - U_1 \delta_{i1}$; we have introduced for later use different reference values for the speed of sound and density. We find it more convenient to work in terms of the stationary coordinate system $\boldsymbol{y}$, in which frame equation (2.5) is

$$\frac{D_1^2 \rho_1'}{D\tau^2} - c_1^2 \frac{\partial^2 \rho_1'}{\partial y_i^2} = \frac{\partial^2 T_{ij}'}{\partial y_i \, \partial y_j}, \tag{2.7}$$

where

$$\frac{D_1}{D\tau} \equiv \frac{\partial}{\partial \tau} + U_1 \frac{\partial}{\partial y_1} \tag{2.8}$$

and the continuity and momentum equations are

$$\left.\begin{aligned} \frac{D_1 \rho'}{D\tau} + \frac{\partial}{\partial y_i}(\rho v_i') &= 0, \\ \frac{D_1}{D\tau}(\rho v_i') + \frac{\partial}{\partial y_j}(T_{ij}' + c_1^2 \rho_1' \delta_{ij}) &= 0. \end{aligned}\right\} \tag{2.9}$$

Suppose that the fluid occupies all space exterior to a surface $\Sigma(\tau)$, which may consist of several closed surfaces. For example $\Sigma(\tau)$ may be the surfaces of several moving bodies. The motion of this surface and/or the heat applied through it generate the jet flow. We choose a region $\nu_1(\tau)$ such that this surface and most of the energetic motion of the fluid are within $\nu_1(\tau)$, while most of the undisturbed fluid lies in $\nu_0(\tau)$, the unbounded region outside $\nu_1(\tau)$. We also suppose that these two regions ν_0 and ν_1 are separated by a surface $S(\tau)$ which moves with the real flow, a situation illustrated in figure 1.

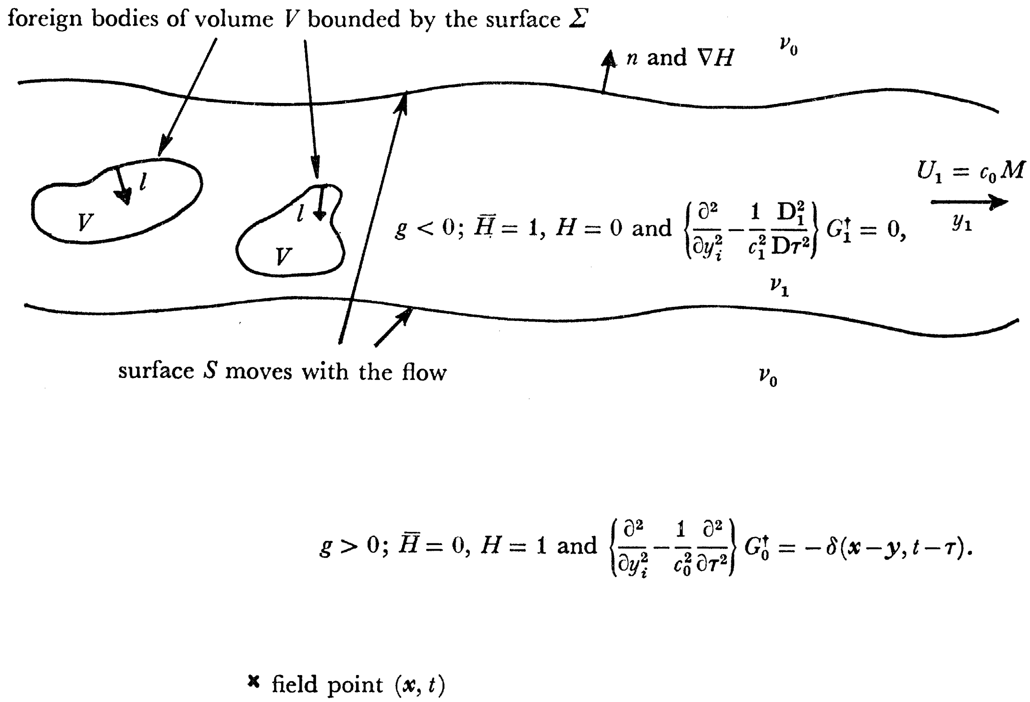

FIGURE 1. A diagram illustrating the geometry and constraints on the Green function.

Let $g(\boldsymbol{y}, \tau)$ be a solution of

$$\frac{\partial g}{\partial \tau} + v_i \frac{\partial g}{\partial y_i} = 0 \tag{2.10}$$

then, $g(\boldsymbol{y}, \tau) =$ constant is the equation of a surface that moves with the fluid and therefore consists of the same fluid particles at all times. It is always possible to choose this solution so that it passes through any prescribed surface at some fixed initial instant. We can therefore normalize g so that $g = 0$ corresponds to $S(\tau)$ and so that g is negative in ν_1, and positive in ν_0. The Heaviside function $H(g)$ is consequently unity in the region ν_0 and zero in ν_1.

Since g is zero on the surface $S(\tau)$ which separates ν_1 from its surrounding ν_0, $\partial H/\partial y_i$ is a vector perpendicular to S, positive in the direction leading from ν_1 into the exterior unbounded fluid in ν_0 and

$$\int_\infty \frac{\partial H}{\partial y_i} K(\boldsymbol{y}, \tau)\, \mathrm{d}^3\boldsymbol{y}\, \mathrm{d}\tau = \int \mathrm{d}\tau \int_{S(\tau)} n_i K\, \mathrm{d}S(\tau)$$

for any function $K(\boldsymbol{y}, \tau)$, where $\boldsymbol{n}$ is in the direction shown in figure 1.

We shall now obtain an exact formula for the density fluctuations which incorporates the idea that if for a sufficiently long time the mean velocity of the fluid in ν_0 is negligible while that in ν_1 does not differ substantially from some constant reference value U_1 which is predominantly in the y_1-direction, it is reasonable to use equations (2.1) and (2.7) respectively to describe their aeroacoustics. Of course, to avoid any extensive linear inhomogeneity of the wave equation we must then choose c_0 and ρ_0 to be characteristic values of the speed of sound and density in ν_0, and c_1 and ρ_1 to be the corresponding values in ν_1. We shall deal exclusively with the exact viscous nonlinear equation of fluid motion and recognize by common observation that the solution is limited by several constraints. First, the jet flow must be driven by an externally applied force distribution acting through the surface Σ. This force is non-zero only for a finite time, which ensures that both the mean flow and sound field are zero if any one of the four space–time co-ordinates tends to infinity. Secondly waves will tend to travel towards infinity from the neighbourhood of the surfaces Σ driving the mean flow.

We choose to work with a reciprocal‡ Green function

$$G^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = H(\boldsymbol{y},\tau)\,G_0^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t)+\bar{H}(\boldsymbol{y},\tau)\,G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t)$$

that has incoming wave behaviour in the variables $\boldsymbol{y}$ and τ, decays as τ becomes large and positive, and is consistent with the equations

$$H\delta(\boldsymbol{x}-\boldsymbol{y},t-\tau) = -H\left\{\frac{\partial^2 G_0^\dagger}{\partial y_i^2}-\frac{1}{c_0^2}\frac{\partial^2 G_0^\dagger}{\partial \tau^2}\right\} \tag{2.11}$$

and

$$\bar{H}\left\{\frac{\partial^2 G_1^\dagger}{\partial y_i^2}-\frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}G_1^\dagger\right\} = 0; \tag{2.12}$$

$\bar{H}$ is written for $1-H$ i.e. $\bar{H}(g) = 1-H(g) = H(-g)$.

At this point $G^\dagger$ is otherwise unrestricted. Notice that we require a weak form of causality: $G^\dagger\to 0$ as $\tau\to\infty$; we do not insist that $G^\dagger$ satisfy the strict causality conditions $G^\dagger = \mathrm{D}_1 G^\dagger/\mathrm{D}\tau = 0$ for $t<\tau$ that are usually imposed on a reciprocal Green function. Indeed it is known (Jones 1973) that in certain model problems, a generalized function bounded at infinity is incompatible with strict causality. We use these equations to determine $H\rho'$ which is a function defined over all space; it is equal to the density fluctuation $\rho-\rho_0$ in ν_0 and zero elsewhere.

$$\begin{aligned} H\rho' &= \int_\infty \rho' H\delta(\boldsymbol{x}-\boldsymbol{y},t-\tau)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau \\ &= -\int_\infty H\rho'\left\{\frac{\partial^2 G_0^\dagger}{\partial y_i^2}-\frac{1}{c_0^2}\frac{\partial^2 G_0^\dagger}{\partial \tau^2}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \end{aligned} \tag{2.13}$$

The result follows from equation (2.11) and the definition of the delta function. The integration ranges over all the four dimensional $(\boldsymbol{y},\tau)$ space. ρ' is an outgoing and $G_0^\dagger$ an incoming acoustic wave at $|\boldsymbol{y}|$ infinity so that equation (2.13) may be integrated by parts twice to give

$$H\rho' = \int_\infty \frac{G_0^\dagger}{c_0^2}\left\{\frac{\partial^2}{\partial\tau^2}-c_0^2\frac{\partial^2}{\partial y_i^2}\right\}(H\rho')\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau, \tag{2.14}$$

an equation that may otherwise be obtained by noting that the real density perturbation, ρ', vanishes at $(|\boldsymbol{y}|,|\tau|)$ infinity.

It is a straightforward matter to develop equation (2.1) by multiplication with H, the transfer of H through the differential operators and the use of equations (2.3), into the form

$$\left\{\frac{\partial^2}{\partial\tau^2}-c_0^2\frac{\partial^2}{\partial y_i^2}\right\}(H\rho') = \frac{\partial^2(HT_{ij})}{\partial y_i\,\partial y_j}-\frac{\partial}{\partial y_j}\left(p_{ij}\frac{\partial H}{\partial y_i}\right)+\rho_0\frac{\partial}{\partial\tau}\left(v_i\frac{\partial H}{\partial y_i}\right), \tag{2.15}$$

‡ The reciprocal Green function is the usual Green function in reverse time, $G^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = G(\boldsymbol{y},-\tau|\boldsymbol{x},-t)$.

which is essentially eqn (2.8) of Ffowcs Williams & Hawkings's (1969) paper. This equation can now be used in (2.14) to give

$$c_0^2 H\rho'(\boldsymbol{x},t) = \int_\infty G_0^\dagger \left\{ \frac{\partial^2 (HT_{ij})}{\partial y_i \partial y_j} - \frac{\partial}{\partial y_j}\left(p_{ij}\frac{\partial H}{\partial y_i}\right) + \rho_0 \frac{\partial}{\partial \tau}\left(v_i \frac{\partial H}{\partial y_i}\right)\right\} \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.16}$$

Here we restrict our attention to the case where all the differential operators can be transferred by partial integration onto the Green function which vanishes on the jet at infinity. The more general situation where the Green function is only bounded within the jet at infinity is considered later. The resulting equation for $H\rho'$ is

$$c_0^2 H\rho'(\boldsymbol{x},t) = \int_\infty \left\{ HT_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i \partial y_j} + p_{ij}\frac{\partial H}{\partial y_i}\frac{\partial G_0^\dagger}{\partial y_j} - \rho_0 v_i \frac{\partial H}{\partial y_i}\frac{\partial G_0^\dagger}{\partial \tau}\right\} \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.17}$$

Next we consider the fluid in the region v_1. The Navier–Stokes equation (2.5) is, of course, only valid in the region exterior to the surface Σ. We therefore introduce H_V a Heaviside function that is unity in $V(\tau)$ the region within the surface $\Sigma(\tau)$, and zero elsewhere. Then if $\Sigma(\tau)$ is impermeable $\partial H_V/\partial\tau + v_i\,\partial H_V/\partial y_i = 0$. Also

$$\int_\infty (\partial H_V/\partial y_i)\, K(\boldsymbol{y},\tau)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau = \int \mathrm{d}\tau \int_{\Sigma(\tau)} l_i K(\boldsymbol{y},\tau)\,\mathrm{d}\Sigma$$

for any function $K(\boldsymbol{y},\tau)$, where $\boldsymbol{l}$ is the normal to the surface $\Sigma(\tau)$ in the direction shown in figure 1. $\bar{H}-H_V$ is non-zero only for points in the fluid in v_1, where (2.5) is valid, and a direct repetition of the argument leading from (2.1) to (2.15), with (2.5) replacing (2.1) and $\bar{H}-H_V$ instead of H, shows that

$$\left\{\frac{\partial^2}{\partial\tau^2}\bigg|_{y'} - c_1^2\frac{\partial^2}{\partial y_i^2}\right\}(\bar{H}-H_V)\,\rho_1' = \frac{\partial^2}{\partial y_i \partial y_j}\{T'_{ij}(\bar{H}-H_V)\} - \frac{\partial}{\partial y_j}\left\{p_{ij}\left(\frac{\partial \bar{H}}{\partial y_i} - \frac{\partial H_V}{\partial y_i}\right)\right\} + \rho_1 \frac{\partial}{\partial\tau}\bigg|_{y'}\left\{v_i'\left(\frac{\partial \bar{H}}{\partial y_i} - \frac{\partial H_V}{\partial y_i}\right)\right\} \tag{2.18}$$

where $\boldsymbol{y}'$ is related to $(\boldsymbol{y},\tau)$ through equation (2.4).

We multiply equation (2.12) by $(\bar{H}-H_V)\,\rho_1'(\boldsymbol{y},\tau)$ and integrate over all space.

$$0 = \int_\infty (\bar{H}-H_V)\,\rho_1'\left\{\frac{\partial^2 G_1^\dagger}{\partial y_i^2} - \frac{1}{c_1^2}\frac{\mathrm{D}_1^2 G_1^\dagger}{\mathrm{D}\tau^2}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.19}$$

The differential operators can again be transferred to the field quantity $(\bar{H}-H_V)\,\rho_1'$ by integration by parts, because $(\bar{H}-H_V)\,G^\dagger$ vanishes at infinity together with its space–time gradient.

$$0 = \int_\infty G_1^\dagger\left\{\frac{\partial^2}{\partial y_i^2} - \frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}\right\}(\bar{H}-H_V)\,\rho_1'\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau$$
$$= \int_\infty G_1^\dagger\left\{\frac{\partial^2}{\partial\tau^2}\bigg|_{y'} - c_1^2\frac{\partial^2}{\partial y_i^2}\right\}(\bar{H}-H_V)\,\rho_1'\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.20}$$

We now substitute (2.18) into (2.20) and rearrange by partial integration; end point contributions are again guaranteed zero by the requirement that $(\bar{H}-H_V)\,G_1^\dagger$ vanish as any one of the variables tends to ± infinity.

$$0 = \int_\infty \left\{(\bar{H}-H_V)\,T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i \partial y_j} + p_{ij}\frac{\partial G_1^\dagger}{\partial y_j}\left(\frac{\partial \bar{H}}{\partial y_i} - \frac{\partial H_V}{\partial y_i}\right) - \rho_1\frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau}v_i'\left(\frac{\partial \bar{H}}{\partial y_i} - \frac{\partial H_V}{\partial y_i}\right)\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.21}$$

Now this equation is multiplied by an arbitrary function of $(\boldsymbol{x}, t)$ or even a linear operator in $(\boldsymbol{x}, t)$, which we denote by β, and the result added to (2.17) to obtain the exact equation

$$c_0^2 H\rho'(\boldsymbol{x}, t) = \int_\infty \left\{ HT_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i \partial y_j} + \beta(\bar{H} - H_V)\, T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i \partial y_j} - \beta p_{ij}\frac{\partial H_V}{\partial y_i}\frac{\partial G_1^\dagger}{\partial y_j} + \beta\rho_1 \frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau} v'_i \frac{\partial H_V}{\partial y_i}\right\} \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau + \Phi(\boldsymbol{x}, t), \tag{2.22}$$

where
$$\Phi(\boldsymbol{x}, t) = \int_\infty \left\{ \frac{\partial \bar{H}}{\partial y_i} p_{ij}\left(\beta\frac{\partial G_1^\dagger}{\partial y_j} - \frac{\partial G_0^\dagger}{\partial y_j}\right) + \rho_0 v_i \frac{\partial \bar{H}}{\partial y_i}\frac{\partial G_0^\dagger}{\partial \tau} - \beta\rho_1 v'_i \frac{\partial \bar{H}}{\partial y_i}\frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau} \right\} \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.23}$$

We have assumed that there is no part of the surface $\Sigma(\tau)$ in the space $\nu_0(\tau)$ and that it does not meet $S(\tau)$, the surface where $\partial\bar{H}/\partial y_i$ is non-zero. The first two terms in equation (2.22) represent sound generation by volume quadrupole sources, the third and fourth terms the field generated by the force and velocity distributions on $\Sigma(\tau)$. $\Phi(\boldsymbol{x}, t)$ represents surface terms on $S(\tau)$, whose strength is linear in the field variable. It is well known that such linear surface sources can often appear to be much more efficient generators of sound than they actually are – especially when one does not know the source strength accurately enough to account properly for phase cancellations. Hence, in its present form, equation (2.22) could easily lead one to make erroneously large predictions of the acoustic field generated by the flow. We overcome this difficulty by using equation (2.10) to rewrite $v'_i\,\partial\bar{H}/\partial y_i$ as $-\mathrm{D}_1\bar{H}/\mathrm{D}\tau$. The last term in (2.23) may be integrated by parts to obtain

$$-\beta\rho_1 \int_\infty v'_i \frac{\partial \bar{H}}{\partial y_i}\frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau = -\beta\rho_1 \int_\infty \bar{H}\frac{\mathrm{D}_1^2 G_1^\dagger}{\mathrm{D}\tau^2}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau.$$

We now introduce a function $\Gamma(\boldsymbol{y}, \tau)$ defined in ν_1 by

$$\bar{H}\rho_0\,\partial^2\Gamma/\partial\tau^2 = \bar{H}\beta\rho_1\,\mathrm{D}_1^2 G_1^\dagger/\mathrm{D}\tau^2, \quad \bar{H}\,\partial\Gamma/\partial\tau \to 0 \quad \text{as} \quad |\tau| \to \infty,$$

so that we can write

$$-\beta\rho_1 \int_\infty \bar{H}\frac{\mathrm{D}_1^2 G_1^\dagger}{\mathrm{D}\tau^2}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau = -\rho_0 \int_\infty \bar{H}\frac{\partial^2\Gamma}{\partial\tau^2}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau$$
$$= -\rho_0 \int_\infty v_i \frac{\partial \bar{H}}{\partial y_i}\frac{\partial \Gamma}{\partial \tau}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau.$$

We find, using $p_{ij} = (p - p_0)\,\delta_{ij} - e_{ij}$, that

$$\Phi(\boldsymbol{x}, t) = \int_\infty \frac{\partial \bar{H}}{\partial y_i}\left\{(p - p_0)\left(\beta\frac{\partial G_1^\dagger}{\partial y_i} - \frac{\partial G_0^\dagger}{\partial y_i}\right) - e_{ij}\left(\beta\frac{\partial G_1^\dagger}{\partial y_j} - \frac{\partial G_0^\dagger}{\partial y_j}\right) + \rho_0 v_i\left(\frac{\partial G_0^\dagger}{\partial \tau} - \frac{\partial \Gamma}{\partial \tau}\right)\right\} \mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \tag{2.24}$$

$G^\dagger$ and Γ are uniquely determined once we specify two jump conditions across $S(\tau)$. We therefore choose conditions that minimize the misleading linear source term Φ, and impose the jump conditions

$$\partial G_0^\dagger/\partial\tau = \partial\Gamma/\partial\tau \quad \text{and} \quad \partial G_0^\dagger/\partial n = \beta\,\partial G_1^\dagger/\partial n \tag{2.25}$$

for $\boldsymbol{y}$ on all parts of $S(\tau)$ where $v_n \neq 0$, $p_{ij} \neq 0$. Then only the viscous term remains in Φ and

$$H\rho'(\boldsymbol{x}, t) = \frac{1}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_0(\tau)} T_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i \partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau + \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_1(\tau)-V(\tau)} T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i \partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau$$
$$+ \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\Sigma(\tau)} l_i\left(\rho_1 v'_i \frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau} - p_{ij}\frac{\partial G_1^\dagger}{\partial y_i}\right)\mathrm{d}\Sigma\,\mathrm{d}\tau$$
$$- \frac{1}{c_0^2}\int_{-\infty}^{\infty}\int_{S(\tau)} e_j^t \frac{\partial}{\partial y_j}(G_0^\dagger - \beta G_1^\dagger)\,\mathrm{d}S\,\mathrm{d}\tau, \tag{2.26}$$

where $\boldsymbol{e}^t = (e_1^t, e_2^t, e_3^t)$ is the projection of the viscous stress $n_i e_{ij}$ onto $S(\tau)$. This equation is exact and could be modified to include surfaces in ν_0 by the addition of the term,

$$\frac{1}{c_0^2}\int_\infty l_i\left(\rho_0 v_i \frac{\partial G_0^\dagger}{\partial \tau} - p_{ij}\frac{\partial G_0^\dagger}{\partial y_j}\right) \mathrm{d}^2\boldsymbol{y}\,\mathrm{d}\tau.$$

This form of the aerodynamic sound equation is the result of a step that avoids the occurrence of an apparently large linear source term in the flow by properly accounting for the embedding of the sound sources in the region of moving fluid while at the same time preventing the occurrence of the confusing linear surface sources. It applies to any real flow and there are no assumptions made about the shape of the regions ν_0 and ν_1. Moreover the function β and the constants U_1, c_1, c_0, ρ_1 and ρ_0 are to a large degree arbitrary. When $U_1 \to 0$, $\beta \to 1$ and $c_1 \to c_0$ equation (2.26) reduces to the Ffowcs Williams–Hawkings (1969) statement of Lighthill's aerodynamic sound theory. But in the general case the first term in equation (2.26) represents the generation of sound by the usual Lighthill source T_{ij} in the exterior region ν_0 while the second term represents the generation of sound by a stress tensor based on relative velocity and density in the interior region ν_1. The third term describes the sound produced by the action of the surface Σ and the last represents the sound generation due to tangential viscous stresses acting across the surface $S(\tau)$. At the high Reynolds numbers which are usually of interest in aerodynamic sound problems, the latter term should be negligible compared with the remaining inertial terms, and we have therefore succeeded in eliminating the linear surface sources; a requisite for obtaining a good estimate of the sound field. This result is a direct consequence of requiring that the dividing surface $S(\tau)$ move with the flow.

In order to establish the representation (2.26) we have assumed that the influence of a point source at $(\boldsymbol{x}, t)$ does not extend to infinity; we have considered $G^\dagger$ for which $\bar{H}G^\dagger$ decays algebraically for large $|\boldsymbol{y}, \tau|$. This assumption will not be true whenever $(\boldsymbol{x}, t)$ is near the jet and can trigger a neutrally stable free mode of the vortex sheet determining $G^\dagger$. Waves can then propagate along the vortex sheet without a fast decay, and even at infinity, $\bar{H}G^\dagger$ will not be negligible. We can easily extend the analysis to deal with such cases. The details are given in appendix A. In fact the only effect of these modes is to ensure that the vortex sheet sources vanish at infinity.

We find

$$\begin{aligned} H\rho'(\boldsymbol{x},t) = \frac{1}{c_0^2}\iint&\left\{HT_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i\,\partial y_j} + \beta((\bar{H}-H_V)\,T'_{ij} - (\bar{H}_0 - H_{V0})\right.\\ &\left.\times(\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij}))\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau\\ &+\frac{1}{c_0^2}\iint\beta\left\{\rho_1\left(U_1\frac{\partial H_{V0}}{\partial y_1} - \frac{\mathrm{D}_1 H_V}{\mathrm{D}\tau}\right)\frac{\mathrm{D}_1 G_1^\dagger}{\mathrm{D}\tau} - \frac{\partial H_V}{\partial y_i}p_{ij}\frac{\partial G_1^\dagger}{\partial y_j}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau\\ &+\frac{1}{c_0^2}\iint\frac{\partial \bar{H}}{\partial y_i}e_{ij}\frac{\partial}{\partial y_j}(G_0^\dagger - \beta G_1^\dagger)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau, \end{aligned} \tag{2.27}$$

where $\bar{H}_0$ is the Heaviside function which is unity for points within $\nu_1^{(0)}$, the region enclosed by the position of the fluid surface S at $\tau = -\infty$, and zero elsewhere. Similarly H_{V0} is unity within the initial position of the surface Σ.

The integrand decays as any one of the variables tends to infinity because

$$G^\dagger \to 0 \quad \text{as} \quad \tau \to \infty$$

and $$(\bar{H}-H_V)\,T'_{ij} \to (\bar{H}_0-H_{V0})\,(\rho_0 U_1^2\delta_{i1}\delta_{j1}-c_1^2(\rho_0-\rho_1)\,\delta_{ij}),$$

$$\frac{\mathrm{D}_1 H_V}{\mathrm{D}\tau} \to U_1\frac{\partial H_{V0}}{\partial y_1} \quad \text{as} \quad |\boldsymbol{y}|,\ -\tau\to\infty.$$

When $G^{\dagger}$ decays at infinity, (2.27) reduces to the previous representation (2.26).

We now simplify this expression by recognizing that the sound generated by the surface distributions that excite the flow is usually quite distinct from the jet noise problem associated with turbulence and will at this stage be discarded, but we recognize in doing so that all engine internal noise and any interaction of the field with diffracting surfaces is thereby lost. The problem we are then restricted to is essentially the free turbulence problem of sound generation in the vicinity of a substantial region of uniformly moving fluid. Hence at sufficiently high Reynolds number, the free turbulence aerodynamic sound equation becomes

$$H\rho'(\boldsymbol{x},t) = \frac{1}{c_0^2}\int_{\infty} T^*_{ij}\frac{\partial^2 G^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau - \frac{\beta}{c_0^2}\int_{\infty}\bar{H}_0\{\rho_0 U_1^2\delta_{i1}\delta_{j1}-c_1^2(\rho_0-\rho_1)\,\delta_{ij}\}\frac{\partial^2 G_1^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau \qquad (2.28)$$

where $$T^*_{ij} = \begin{cases} T_{ij} & \text{for} \quad \boldsymbol{y} \text{ in } \nu_0 \\ \beta T'_{ij} & \text{for} \quad \boldsymbol{y} \text{ in } \nu_1. \end{cases}$$

The integration is to be carried out over all four-dimensional space. Thus for high Reynolds number flows, all linear surface sources have been eliminated.

In most real flows the turbulent sound sources will be concentrated around the high mean velocity region. It is then reasonable to choose $\nu_1(\tau)$ so that it coincides with this region and to neglect the source distribution in ν_0. Then equation (2.28) becomes

$$H\rho'(\boldsymbol{x},t) = \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_1(\tau)} T'_{ij}\frac{\partial^2 G_1^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau - \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_1^{(0)}}\{\rho_0 U_1^2\delta_{i1}\delta_{j1}-c_1^2(\rho_0-\rho_1)\,\delta_{ij}\}\frac{\partial^2 G_1^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau \qquad (2.29)$$

and the equation gives the sound field everywhere, except inside the flow itself.

T'_{ij} has a large mean value
$$\{\rho_0 U_1^2\delta_{i1}\delta_{j1}-c_1^2(\rho_0-\rho_1)\,\delta_{ij}\}$$
on the surface S which is now deliberately positioned in the linearly disturbed fluid that surrounds the more energetic flow in $\nu_1(\tau)$, and at any fixed $\boldsymbol{y}$, the integrand in the first volume integral of equation (2.28) varies enormously as the interface drifts *quietly* by, thereby generating the false impression of powerful linear surface sources of the type we have been at such pains to avoid. This possibility is ruled out if we choose to express the first integral in terms of a coordinate system $\boldsymbol{\eta}$ that moves with the individual fluid particles. The transformation from the Eulerian to Lagrangian coordinates has a Jacobian ρ^*/ρ, ρ^* being the density of the fluid particle identified by the coordinate $\boldsymbol{\eta}$ at the time when the Lagrangian and Eulerian coordinates coincide; i.e. at the particular time when $\boldsymbol{\eta} = \boldsymbol{y}$.

Our optimal description of the induced field is then provided by the Lagrangian form of equation (2.29)

$$H\rho'(\boldsymbol{x},t) = \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_1}\rho^*(\boldsymbol{\eta})\left\{\frac{T'_{ij}}{\rho}\right\}\frac{\partial^2 G_1^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{\eta}\,\mathrm{d}\tau - \frac{\beta}{c_0^2}\int_{-\infty}^{\infty}\int_{\nu_1^{(0)}}\{\rho_0 U_1^2\delta_{i1}\delta_{j1}-c_1^2(\rho_0-\rho_1)\,\delta_{ij}\}\frac{\partial^2 G_1^{\dagger}}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad (2.30)$$

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This is the field generated by a moving distribution of quadrupoles of strength density T'_{ij}/ρ; each moving fluid particle is acoustically equivalent to a quadrupole whose strength per unit mass is

$$v'_i v'_j + p_{ij}/\rho - c_1^2(1-\rho_1/\rho)\,\delta_{ij}. \tag{2.31}$$

The reciprocal Green function $G^\dagger$ which appears in these equations is an incoming wave solution to equations (2.11) and (2.12) which satisfies the two jump conditions (2.25) across S, vanishes as $\tau \to \infty$, and is bounded within ν_1 as $|\boldsymbol{y}|, -\tau \to \infty$.

The function $\beta(\boldsymbol{x}, t)$ is arbitrary, and it is easy to see that the density fluctuation does not depend on the choice of β. Since, if $G^{(1)}$ denotes the solution to the boundary value problem with $\beta = 1$ and $G^\dagger$ denotes a solution for any other value of β, it follows from equations (2.11) and (2.25) that these solutions must be related by

$$G^\dagger = G^{(1)} \quad \text{for} \quad \boldsymbol{y} \text{ in } \nu_0,$$

$$\beta G^\dagger = G^{(1)} \quad \text{for} \quad \boldsymbol{y} \text{ in } \nu_1.$$

Hence when $\boldsymbol{y}$ is in ν_1, $\beta G^\dagger$ is invariant under the choice of β, while for $\boldsymbol{y}$ in ν_0, $G^\dagger$ itself possesses this invariance. Since these are the quantities which actually appear in equation (2.28) we see that $H\rho'(\boldsymbol{x}, t)$ must be independent of β. We have only introduced this function to facilitate the interpretation of the jump conditions (2.25). Although these jump conditions are just sufficient to uniquely determine $G^\dagger$, its calculation presents us with a formidable task since the surface $S(\tau)$, on which the conditions are applied, moves with the actual flow. For real turbulent flows, any such surface will become highly irregular. Indeed it is even difficult at this stage to arrive at a reasonable physical interpretation of $G^\dagger$. Fortunately, for the nearly parallel flows which are of technological interest, it is possible to construct the reciprocal Green function and we shall now demonstrate this to be the Green function appropriate to an instability-free (and therefore only weakly causal) vortex sheet modelling of the flow.

3. Vortex sheet model

Suppose now that S_0, the initial position of S, is a cylindrical surface doubly infinite in the y_1-direction whose generators lie parallel to the mean flow $\boldsymbol{U}$. If the surface $S(\tau)$ lies in a linearly disturbed flow, then, in some sense, it remains close to its initial position S_0, and also v_i, p_{ij} and n_1 are small. We can therefore adopt the procedure used in linearized aerodynamics and 'transfer' the boundary terms in equation (2.24) from $S(\tau)$ to the fixed parallel surface S_0. The problem of determining the Green function is now much more tractable. Since S_0 is a fixed surface, we can easily eliminate Γ from the jump condition (2.25) to obtain

$$\rho_0 \frac{\partial^2 G_0^\dagger}{\partial \tau^2} = \beta \rho_1 \frac{\mathrm{D}_1^2 G_1^\dagger}{\mathrm{D}\tau^2} \quad \text{and} \quad \frac{\partial G_0^\dagger}{\partial n} = \beta \frac{\partial G_1^\dagger}{\partial n} \quad \text{on} \quad S_0. \tag{3.1}$$

Moreover, there is in this problem no characteristic scale for either the time or the distance in the y_1-direction. Hence $G_0^\dagger$ and $G_1^\dagger$ can depend on t, τ, x_1, and y_1 only in the combination $t-\tau$ and $x_1 - y_1$, so that

$$\frac{\partial G_0^\dagger}{\partial \tau} = -\frac{\partial G_0^\dagger}{\partial t}, \quad \frac{\mathrm{D}_1 G_0^\dagger}{\mathrm{D}\tau} = -\left\{\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x_1}\right\} G_0^\dagger \tag{3.2}$$

and the derivatives of $G_1^\dagger$ satisfy the same relation.

We now restrict our attention to the case where the observation point $\boldsymbol{x}$ is in the far field. Then the requirement that $G^{\dagger}$ represent an incoming wave in $\boldsymbol{y}$ and τ ensures that it will represent an outgoing wave in $\boldsymbol{x}$ and t so that

$$\frac{\partial G^{\dagger}}{\partial x_1} = -\frac{1}{c_0}\frac{\partial G^{\dagger}}{\partial \tau}\frac{\partial |\boldsymbol{x}|}{\partial x_1} = -\frac{x_1}{|\boldsymbol{x}|\,c_0}\frac{\partial G^{\dagger}}{\partial t}.$$

Hence
$$\frac{\mathrm{D}_1 G_0^{\dagger}}{\mathrm{D}\tau} = -(1-M_r)\frac{\partial G_0^{\dagger}}{\partial t} = (1-M_r)\frac{\partial G_0^{\dagger}}{\partial \tau},$$

and similarly
$$\frac{\mathrm{D}_1 G_1^{\dagger}}{\mathrm{D}\tau} = (1-M_r)\frac{\partial G_1^{\dagger}}{\partial \tau},$$

where $M_r = Mx_1/|\boldsymbol{x}|$, and $M \equiv U_1/c_0$ is the effective Mach number of the mean flow based on the velocity U_1 in ν_1 and the speed of sound c_0 in ν_0. By using this relation, the boundary conditions (3.1) can be put in the form

$$\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}\frac{\partial G_0^{\dagger}}{\partial n} = \rho_0 B\frac{\partial^2}{\partial \tau^2}\frac{\partial G_1^{\dagger}}{\partial n},$$

$$G_0^{\dagger} = BG_1^{\dagger}, \tag{3.3}$$

where
$$B = \beta(1-M_r)^2\rho_1/\rho_0.$$

But since the final result is independent of our choice of β, we can put

$$1 = \beta(1-M_r)^2\rho_1/\rho_0 \tag{3.4}$$

whereupon the jump conditions (3.1) become more familiar. They are in fact the usual acoustic boundary conditions

$$\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}\frac{\partial G_0^{\dagger}}{\partial n} = \rho_0\frac{\partial^2}{\partial \tau^2}\frac{\partial G_1^{\dagger}}{\partial n} \quad \text{and} \quad G_0^{\dagger} = G_1^{\dagger} \tag{3.5}$$

of continuity of particle displacement and pressure which apply across a linearly disturbed *vortex sheet*

Now let $G(\boldsymbol{x}, t|\boldsymbol{y}, \tau)$ be the generalized function which is an outgoing wave solution (in the variables $\boldsymbol{x}, t$) to the equations

$$\left\{\frac{\partial^2}{\partial x_i^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right\} G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = 0 \quad \text{if} \quad \boldsymbol{x} \text{ is in } \nu_0^{(0)} \text{ and } \boldsymbol{y} \text{ in } \nu_1^{(0)}$$

$$\left\{\frac{\partial^2}{\partial x_i^2} - \frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}\right\} G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = -\delta(\boldsymbol{x}-\boldsymbol{y}, t-\tau) \quad \text{if} \quad \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are in } \nu_1^{(0)}, \tag{3.6}$$

and satisfies the jump conditions (3.5). In this equation $\nu_0^{(0)}$ and $\nu_1^{(0)}$ denote the fixed regions bounded by the cylindrical surface S_0 and

$$\frac{\mathrm{D}_1}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x_1}.$$

G simply represents the sound field due to an acoustic point source located in an infinite 'instability-free' cylindrical jet with a uniform velocity profile. It is shown in appendix B that it is related to the reciprocal Green function, $G^{\dagger}$, by

$$\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = \rho_0\frac{\partial^2 G_1^{\dagger}}{\partial t^2}(\boldsymbol{y}, \tau|\boldsymbol{x}, t) \tag{3.7}$$

for $\boldsymbol{x}$ in the radiation field in $\nu_0^{(0)}$, $\boldsymbol{y}$ in $\nu_1^{(0)}$; this is just

$$\rho_1(1-M_r)^2 G(\boldsymbol{x},t|\boldsymbol{y},\tau) = \rho_0 G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t). \tag{3.8}$$

Thus the Green function which appears in the acoustic analogy equation (2.30) is very nearly the usual acoustic Green function for an infinite cylindrical jet with slug flow velocity profile and a vortex sheet boundary discontinuity. It differs from it because it is required to satisfy a boundedness and a weak causality condition rather than strict causality.

The appearance of this weakly causal Green function in (2.30) means that the information about the turbulence propagates into the far field at a speed greater than c_0. We believe that this is reasonable. The representation (2.30) describes the way in which the sound field depends on the turbulence, but it is not the simple relation between an observed sound and its 'source'. In an unstable jet, sound waves can *cause* turbulence, and then later that turbulence can become a source of sound. Equation (2.30) must therefore describe the more complicated interdependence of the turbulence and the sound field. It is difficult to see how the strictly causal Green function could be used in such a representation because of problems concerned with the convergence of the integrals. The strictly causal Green function would of course, describe the sound field of a point source near a real vortex sheet, but it is the weakly causal Green function that is relevant to the jet noise problem.

4. The circular cylindrical jet

In order to illustrate the vortex sheet analogy, we now consider a simple geometry for which it is possible to obtain $G^\dagger$ explicitly. We investigate a turbulent round jet of radius a, with mean flow U_1 in the 1-direction, emitting sound into a linearly disturbed fluid. It is convenient to introduce cylindrical coordinates, and we write

$$\boldsymbol{x} = (R, \Phi, x_1), \quad \boldsymbol{y} = (\sigma, \phi, y_1)$$

then from (2.30)

$$c_0^2 H\rho'(\boldsymbol{x},t) = \frac{\rho_0}{\rho_1(1-M_r)^2}\int \mathrm{d}\tau \left\{ \int_{\nu_1} \frac{\rho^* T'_{ij}}{\rho} \frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{\eta} - \int_{\nu_1^{(0)}} (\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij}) \frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y} \right\}, \tag{4.1}$$

where

$$\left\{\frac{\partial^2}{\partial y_i^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial \tau^2}\right\} G_0^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = -\delta(\boldsymbol{x}-\boldsymbol{y}, t-\tau) \quad \text{for} \quad \sigma > a$$

$$\left\{\frac{\partial^2}{\partial y_i^2} - \frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}\right\} G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = 0 \quad \text{for} \quad \sigma < a \tag{4.2}$$

together with the vortex sheet jump conditions

$$G_0^\dagger = G_1^\dagger, \quad \rho_1 \frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}\frac{\partial G_0^\dagger}{\partial \sigma} = \rho_0 \frac{\partial^2}{\partial \tau^2}\frac{\partial G_1^\dagger}{\partial \sigma} \quad \text{on} \quad \sigma = a. \tag{4.3}$$

However $G^\dagger$ differs from the vortex sheet Green function found by Morgan (1975) in that it is devoid of instabilities because

$$\bar{H}G^\dagger \sim O(1) \quad \text{as} \quad |\boldsymbol{y},\tau| \to \infty.$$

$G^\dagger$ cannot therefore be strictly causal. This system of equations can be solved most simply by taking Fourier transforms in y_1, τ and ϕ. (Actually the angular dependence is expanded as a Fourier series.) If we denote the transform of $G_1^\dagger$ by $\bar{G}_1$ then

$$\bar{G}_1(\sigma,k,n,\omega|\boldsymbol{x},t) = \int_{\phi=0}^{2\pi}\int_{y_1=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t)\exp\{-i(ky_1+\omega\tau+n\phi)\}\,\mathrm{d}y_1\,\mathrm{d}\tau\,\mathrm{d}\phi$$

and $$G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\bar{G}_1(\sigma,k,n,\omega|\boldsymbol{x},t)\exp\{\mathrm{i}(ky_1+\omega\tau+n\phi)\}\,\mathrm{d}k\,\mathrm{d}\omega.$$

$\bar{G}_0(\sigma,k,n,\omega|\boldsymbol{x},t)$ satisfies

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}\bar{G}_0}{\mathrm{d}\sigma}\right)+\left(\frac{\omega^2}{c_0^2}-k^2-\frac{n^2}{\sigma^2}\right)\bar{G}_0 = -\frac{\delta(\sigma-R)}{\sigma}\exp\{-\mathrm{i}(kx_1+\omega t+n\Phi)\}, \tag{4.4}$$

which has the solution

$$\bar{G}_0 = -\frac{\exp\{-\mathrm{i}(kx_1+\omega t+n\Phi)\}}{RW(f_1,f_2)}\begin{cases} f_1(\sigma)f_2(R) & \text{for} \quad \sigma \leqslant R \\ f_1(R)f_2(\sigma) & \text{for} \quad \sigma \geqslant R. \end{cases} \tag{4.5}$$

f_1 and f_2 are two Bessel functions chosen to satisfy the boundary conditions and W is their Wronksian evaluated at R (see for example, Morse & Feshbach 1953, p. 826). The general form of f_1 is $f_1(\sigma) = J_n(\gamma_0\sigma)+BH_n^{(1)}(\gamma_0\sigma)$ where $\gamma_0^2 = \omega^2/c_0^2-k^2$. $G_0^\dagger$ behaves like an incoming wave at infinity in the $(\boldsymbol{y},\tau)$ coordinates so $f_2(\sigma) = H_n^{(1)}(\gamma_0\sigma)$, where the root of γ_0 is chosen so that when real, γ_0 has the sign of ω. When γ_0 is complex its imaginary part is positive so that $G_0^\dagger$ is bounded at infinity. Both these conditions are satisfied by the Riemann sheet $\operatorname{Im}\gamma_0 > 0$ with a branch cut along $\operatorname{Im}\gamma_0 = 0$ as shown in figure 2.

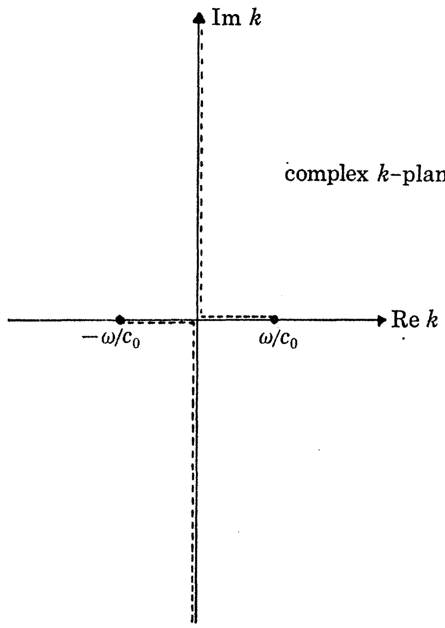

FIGURE 2. The position of the branch cuts.

We note that $W(f_1,f_2) = W(J_n(\gamma_0 R), H_n^{(1)}(\gamma_0 R))$ is $2\mathrm{i}/\pi R$. The Fourier transform of the second equation in (4.2) gives

$$\frac{1}{\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\sigma\frac{\mathrm{d}\bar{G}_1}{\mathrm{d}\sigma}\right)+\left(\frac{(\omega+U_1k)^2}{c_1^2}-k^2-\frac{n^2}{\sigma^2}\right)\bar{G}_1 = 0 \quad \text{in} \quad \sigma < a. \tag{4.6}$$

$\bar{G}_1$ must be finite at $\sigma = 0$ and hence

$$\bar{G}_1(\sigma,k,n,\omega|\boldsymbol{x},t) = CJ_n(\gamma\sigma), \quad \text{where} \quad \gamma = \{(\omega+U_1k)^2/c_1^2-k^2\}^{\frac{1}{2}}.$$

The constant C can be determined by applying the jump condition (4.3) at $\sigma = a$ to obtain

$$\bar{G}_1(\sigma, k, n, \omega|\boldsymbol{x}, t) = \frac{\rho_1(\omega+U_1 k)^2 H_n^{(1)}(\gamma_0 R)\, J_n(\gamma\sigma)}{F_n(\omega, k)} \exp\{-\mathrm{i}(kx_1+\omega t+n\Phi)\}, \tag{4.7}$$

where $\quad F_n(\omega, k) = a\{\rho_0\omega^2\gamma J_n'(\gamma a)\, H_n^{(1)}(\gamma_0 a) - \rho_1(\omega+U_1 k)^2\gamma_0 J_n(\gamma a)\, H_n^{(1)\prime}(\gamma_0 a)\},$

and the prime denotes differentiation with respect to the argument.

Inversion of the Fourier transform gives

$$G_1^\dagger(\boldsymbol{y}, \tau|\boldsymbol{x}, t) = \frac{1}{(2\pi)^3}\iint \sum_{n=-\infty}^{\infty} \frac{\rho_1(\omega+U_1 k)^2 H_n^{(1)}(\gamma_0 R) J_n(\gamma\sigma)}{F_n(\omega, k)} \times \exp \mathrm{i}\{k(y_1-x_1)+\omega(\tau-t)+n(\phi-\Phi)\}\,\mathrm{d}k\,\mathrm{d}\omega, \tag{4.8}$$

where the ω-integral is to be taken along the weakly causal contour which lies above the poles and branch cuts on the real axis in order to satisfy the condition $G^\dagger \to 0$ as $\tau\to\infty$. We now introduce θ, the angle between the distant observation point and the direction of flow, defined by $R = |\boldsymbol{x}|\sin\theta$ and $x_1 = |\boldsymbol{x}|\cos\theta$. Then $H_n^{(1)}(\gamma_0 R) = H_n^{(1)}(\gamma_0\sin\theta|\boldsymbol{x}|)$ and for $\boldsymbol{x}$ in the far field we can expand $H_n^{(1)}$ by its asymptotic form for large arguments and write

$$H_n^{(1)}(\gamma_0 R) \sim \left(\frac{2}{\pi\gamma_0\sin\theta|\boldsymbol{x}|}\right)^{\frac{1}{2}} \exp \mathrm{i}\{\gamma_0\sin\theta|\boldsymbol{x}| - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi\} \quad \text{if} \quad \gamma_0\sin\theta \neq 0.$$

Hence

$$G_1^\dagger(\boldsymbol{y}, \tau|\boldsymbol{x}, t) \sim \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \exp \mathrm{i}\{\omega(\tau-t)+n(\phi-\Phi-\tfrac{1}{2}\pi)-\tfrac{1}{4}\pi\}\,\mathrm{d}\omega \times \int\left(\frac{2}{\pi\gamma_0\sin\theta|\boldsymbol{x}|}\right)^{\frac{1}{2}} \frac{\rho_1(\omega+U_1k)^2 J_n(\gamma\sigma)}{F_n(\omega, k)} \exp\{\mathrm{i}ky_1+|\boldsymbol{x}|\, h(k,\omega)\}\,\mathrm{d}k. \tag{4.9}$$

For large $|\boldsymbol{x}|$, the k-integral is in a form suitable for evaluation by the method of stationary phase. The stationary point of $h(k,\omega) = -\mathrm{i}k\cos\theta + \mathrm{i}\gamma_0\sin\theta$ on the Riemann sheet $\mathrm{Im}\,\gamma_0 > 0$ occurs at $k = -\omega\cos\theta/c_0$, and the path of steepest descent is therefore a curve C in the complex k-plane defined by $-\mathrm{i}k\cos\theta+\mathrm{i}\gamma_0\sin\theta-\mathrm{i}\omega/c_0 = -u^2$ where u is real on C. A sketch of the curve C is shown in figure 3.

We now make the usual approximation in the method of steepest descents and write

$$\int_C \left(\frac{2}{\pi\gamma_0\sin\theta|\boldsymbol{x}|}\right)^{\frac{1}{2}} \frac{\rho_1(\omega+U_1k)^2}{F_n(\omega,k)} J_n(\gamma\sigma)\exp\{\mathrm{i}ky_1+|\boldsymbol{x}|\,h(k,\omega)\}\,\mathrm{d}k = \frac{2\rho_1\omega^2(1-M_r)^2 J_n(\gamma\sigma)}{|\boldsymbol{x}|\,F_n(\omega, -\omega\cos\theta/c_0)} \exp \mathrm{i}\{\omega(|\boldsymbol{x}|-y_1\cos\theta)/c_0 - \tfrac{1}{4}\pi\}, \tag{4.10}$$

where now $\quad \gamma_0 = \omega\sin\theta/c_0, \quad \gamma = \omega\left\{\dfrac{(1-M_r)^2}{c_1^2} - \dfrac{\cos^2\theta}{c_0^2}\right\}^{\frac{1}{2}}$

and $\quad M_r = U_1\cos\theta/c_0.$

$F_n(\omega, k)$ has no zeros for ω and k real, with $|k|c_0 < |\omega|$, because for this range of k $\gamma J_n'(\gamma a)/J_n(\gamma a)$ is real and the imaginary part of $\gamma_0 H_n^{(1)\prime}(\gamma_0 a)/H_n^{(1)}(\gamma_0 a)$ is non-zero. It therefore follows that when we deform the integration path from the real k-axis on to the contour C, the contribution from any pole crossed over is negligible. Morgan (1975) argues that the contribution from the branch cut integral is $O(|\boldsymbol{x}|^{-\frac{3}{2}})$ in which case, for $\boldsymbol{x}$ in the far field, we can neglect it giving

$$G_1^\dagger(\boldsymbol{y}, \tau|\boldsymbol{x}, t) = \frac{\rho_1(1-M_r)^2}{4\pi^3|\boldsymbol{x}|\mathrm{i}}\int\sum_{n=-\infty}^{\infty}\frac{\omega^2 J_n(\gamma\sigma)}{F_n(\omega, -\omega\cos\theta/c_0)}\exp \mathrm{i}\{\omega(\tau-t^*)+n(\phi-\Phi-\tfrac{1}{2}\pi)\}\,\mathrm{d}\omega, \tag{4.11}$$

where t^* is the retarded time appropriate to waves travelling to $\boldsymbol{x}$ at speed c_0,

$$t^* = t - (|\boldsymbol{x}| - y_1 \cos\theta)/c_0.$$

By a similar argument we find

$$G_0^\dagger(\boldsymbol{y}, \tau | \boldsymbol{x}, t) = \frac{1}{8\pi^2|\boldsymbol{x}|}\int \sum_{n=-\infty}^{\infty}\left\{J_n(\gamma_0\sigma) + \frac{B_n(\omega)\, H_n^{(1)}(\gamma_0\sigma)}{F_n(\omega, -\omega\cos\theta/c_0)}\right\}\exp \mathrm{i}\{\omega(\tau - t^*) + n(\phi - \Phi - \tfrac{1}{2}\pi)\}\,\mathrm{d}\omega \tag{4.12}$$

with $$B_n(\omega) = a\omega^2\{\rho_1(1-M_r)^2\gamma_0 J_n'(\gamma_0 a)\, J_n(\gamma a) - \rho_0\gamma J_n(\gamma_0 a)\, J_n'(\gamma a)\}.$$

$F_n(\omega, -\omega\cos\theta/c_0)$ has no zeros for real ω and so a neutrally stable mode of the vortex sheet jet cannot be excited when $\boldsymbol{x}$ is in the far field.

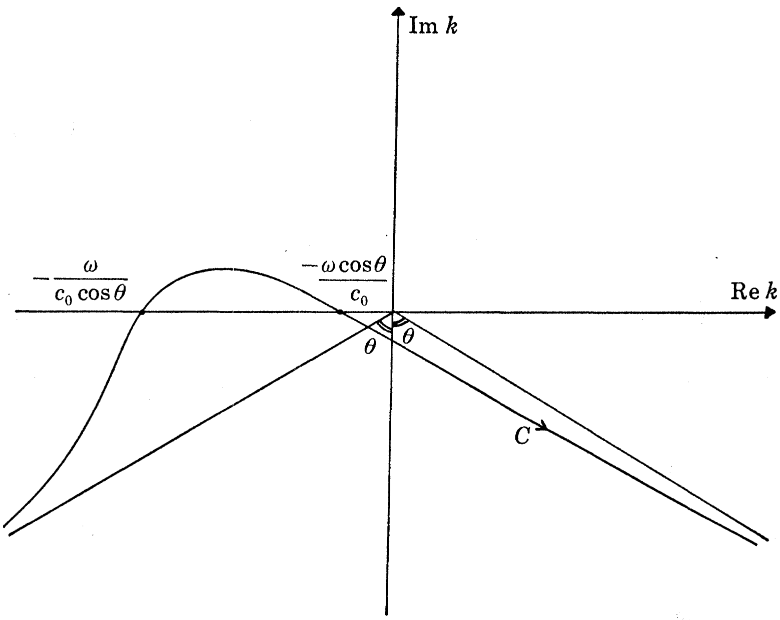

FIGURE 3. The position of the curve of stationary phase.

When $\tau - t^* - a\sin\theta/c_0$ is positive, we can evaluate $G_1^\dagger$ by closing the ω-contour with a large semi-circle in the upper half plane;

$$G_1^\dagger = \sum_{n=-\infty}^{\infty}\sum_j Q_j \exp\{\mathrm{i}\omega_j(\tau - t^* - a\sin\theta/c_0)\} \tag{4.13}$$

where the sum is over the poles, ω_j, in the upper half plane, and $Q_j \exp\{\mathrm{i}\omega_j(\tau - t^* - a\sin\theta/c_0)\}$ are the residues at ω_j.

When the vortex sheet is unstable, and such poles do exist, $G_1^\dagger$ can be non-zero for

$$\tau - t^* - a\sin\theta/c_0 > 0$$

and is therefore not strictly causal.

We define K by Ka/c_0 = minimum $(\mathrm{Im}\,\omega_j)$. From the form of F_n, we see that K is a function of $\cos\theta$, M, n and the ratios c_1/c_0, ρ_1/ρ_0. Numerical calculations give $K = 0.8$ at a Mach number $M = 2$, $\cos\theta = \frac{2}{3}$, $\rho_1 = \rho_0$ and $c_1 = c_0$ for the symmetric mode $n = 0$.

There is a 'precursor' ahead of the wave front, that is at a point $(\boldsymbol{x}, t)$ such that

$$t^* < \tau - a\sin\theta/c_0.$$

The precursor decays exponentially ahead of this front, and will in fact be negligible for $t^* < \tau - a\sin\theta/c_0 - Ka/c_0$. Therefore, for this example of the symmetric mode, sound will only be heard up to a jet diameter or so ahead of the front.

We now restrict our detailed analysis to the compact case which is algebraically more straightforward. By 'compact' we mean a jet with diameter small in comparison with the acoustic wavelength.

From equation (4.1)

$$H\rho'(\boldsymbol{x},t) = \frac{\rho_0}{\rho_1(1-M_r)^2 c_0^2}\int \{\bar{H}T'_{ij} - \bar{H}_0(\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})\}\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \quad (4.14)$$

Parseval's theorem gives

$$\int \{\bar{H}T'_{ij} - \bar{H}_0(\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})\}\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau$$
$$= \frac{1}{2\pi}\int \tilde{T}_{ij}(\boldsymbol{y}, -\omega)\,\frac{\partial^2 G_1}{\partial y_i\,\partial y_j}(\boldsymbol{y},\omega)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\omega, \quad (4.15)$$

where $\tilde{T}_{ij}(\boldsymbol{y},\omega)$ is the Fourier transform of $\bar{H}T'_{ij} - \bar{H}_0(\rho_0 U_1^2\delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})$ with respect to time, and $G_1(\boldsymbol{y},\omega)$ is the transform of $G_1^\dagger(\boldsymbol{y},\tau)$. Hence (4.14) may be rewritten as

$$H\rho'(\boldsymbol{x},t) = \frac{\rho_0}{\rho_1(1-M_r)^2 c_0^2\, 2\pi}\int \tilde{T}_{ij}(\boldsymbol{y}, -\omega)\,\frac{\partial^2 G_1}{\partial y_i\,\partial y_j}(\boldsymbol{y},\omega)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\omega. \quad (4.16)$$

For a compact jet $\tilde{T}_{ij}(\boldsymbol{y},\omega)$ is zero unless $|\omega|a \ll c_0$ and we can replace $G_1(\boldsymbol{y},\omega)$ in (4.16) by $G_1^c(\boldsymbol{y},\omega)$, its low frequency asymptotic form.

Taking transforms of (4.11) we find

$$G_1(\boldsymbol{y},\omega) = \frac{\rho_1(1-M_r)^2\omega^2}{2\pi^2|\boldsymbol{x}|\mathrm{i}}\sum_{n=-\infty}^{\infty}\frac{J_n(\gamma\sigma)}{F_n(\omega, -\omega\cos\theta/c_0)}\exp\mathrm{i}\{-\omega t^* + n(\phi - \Phi - \tfrac{1}{2}\pi)\}. \quad (4.17)$$

and when $\omega a/c_0 \ll 1$ we can use the expansions for Bessel functions of small argument to obtain

$$G_1^c(\boldsymbol{y},\omega) = \frac{1}{|\boldsymbol{x}|}\{a_0(\omega)\,(1-\tfrac{1}{4}(\gamma\sigma)^2+\ldots) - a_1(\omega)\,\mathrm{i}\gamma\sigma\cos(\phi-\Phi)$$
$$- a_2(\omega)\,\tfrac{1}{4}\gamma^2\sigma^2\cos 2(\phi-\Phi) + O(\epsilon^3)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*}, \quad (4.18)$$

where $\epsilon = \omega a/c_0$ and $a_n(\omega)$ is the low frequency expansion of

$$\frac{\rho_1(1-M_r)^2\omega^2}{2\pi^2\mathrm{i}F_n(\omega, -\omega\cos\theta/c_0)}.$$

In fact

$$a_n(\omega) = a_{-n}(\omega) = \frac{\rho_1(1-M_r)^2\gamma_0^n}{2\pi\gamma^n\{\rho_0+\rho_1(1-M_r)^2+O(\rho_0\epsilon^2, \rho_1\epsilon^2, \rho_0\epsilon^{2n}\ln\epsilon, \rho_1\epsilon^{2n}\ln\epsilon)\}} \quad \text{for} \quad n \geqslant 1$$

and

$$a_0(\omega) = \frac{\rho_1(1-M_r)^2}{2\pi\{2\rho_1(1-M_r)^2 + a^2\ln(\gamma_0 a)\,(\rho_0\gamma^2 - \rho_1(1-M_r)^2\gamma_0^2) + O(\rho_0\epsilon^2, \rho_1\epsilon^2)\}}.$$

We choose the $\boldsymbol{y}$-axes so that $y_2 = \sigma\cos\phi$, $y_3 = \sigma\sin\phi$, then

$$G_1^c(\boldsymbol{y},\omega) = \frac{1}{|\boldsymbol{x}|}\{a_0(\omega)\,(1-\tfrac{1}{4}\gamma^2(y_2^2+y_3^2)\ldots) - \mathrm{i}\gamma a_1(\omega)\,(y_2\cos\Phi + y_3\sin\Phi)$$
$$-\tfrac{1}{4}\gamma^2 a_2(\omega)\,((y_2^2-y_3^2)\cos 2\Phi + 2y_2 y_3\sin 2\Phi)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*}, \quad (4.19)$$

and the derivatives of $G_1^c(\boldsymbol{y},\omega)$ with respect to $\boldsymbol{y}$ can easily be evaluated;

$$\left.\begin{aligned}
\frac{\partial^2 G_1^c}{\partial y_1^2}(\boldsymbol{y},\omega) &= -\frac{\omega^2\cos^2\theta}{|\boldsymbol{x}|c_0^2}\{a_0(\omega)+O(\epsilon)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*},\\
\frac{\partial^2 G_1^c}{\partial y_1\,\partial y_2}(\boldsymbol{y},\omega) &= -\frac{\omega^2\cos\theta}{|\boldsymbol{x}|c_0^2}\{\alpha\cos\Phi a_1(\omega)+O(\epsilon a_0)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*},\\
\frac{\partial^2 G_1^c}{\partial y_2^2}(\boldsymbol{y},\omega) &= -\frac{\omega^2}{|\boldsymbol{x}|c_0^2}\{\tfrac{1}{2}\alpha^2(a_0(\omega)+a_2(\omega)\cos 2\Phi)+O(\epsilon^2a_0,\epsilon a_1)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*},\\
\frac{\partial^2 G_1^c}{\partial y_2\,\partial y_3}(\boldsymbol{y},\omega) &= -\frac{\omega^2}{|\boldsymbol{x}|c_0^2}\{\tfrac{1}{2}\alpha^2\sin 2\Phi\, a_2(\omega)+O(\epsilon^2a_0,\epsilon a_1)\}\,\mathrm{e}^{-\mathrm{i}\omega t^*},
\end{aligned}\right\} \tag{4.20}$$

where $\alpha = \{(1-M_r)^2c_0^2/c_1^2-\cos^2\theta\}^{\frac{1}{2}}$.

Now that the derivatives of $G_1^c(\boldsymbol{y},\omega)$ have been determined we can use equation (4.16) to estimate the sound field. Alternatively the sound field may be described by (4.14) with $G_1^\dagger(\boldsymbol{y},\tau)$ replaced by $G_1^c(\boldsymbol{y},\tau)$, where

$$G_1^c(\boldsymbol{y},\tau) = \frac{1}{2\pi}\int G_1^c(\boldsymbol{y},\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega. \tag{4.21}$$

Away from both the mean flow Mach angle and the jet axis we can neglect the second term in the denominator of a_0 in comparison with the first, provided only that the mean jet density is greater than, or of the same order as, the density of the ambient fluid. Then

$$\frac{\partial^2 G_1^c}{\partial y_i\,\partial y_j}(\boldsymbol{y},\omega) = -\frac{\omega^2\mathrm{D}_{ij}}{4\pi|\boldsymbol{x}|c_0^2}\frac{\rho_1(1-M_r)^2}{\rho_0}\,\mathrm{e}^{-\mathrm{i}\omega t^*}, \tag{4.22}$$

where D_{ij} is a direction factor;

$$\mathrm{D}_{11} = \frac{\rho_0}{\rho_1(1-M_r)^2}\frac{x_1^2}{|\boldsymbol{x}|^2},$$

$$\mathrm{D}_{22} = \left\{\tfrac{1}{2}\alpha^2+\frac{\rho_1(1-M_r)^2}{\rho_0+\rho_1(1-M_r)^2}\frac{(x_2^2-x_3^2)}{|\boldsymbol{x}|^2}\right\}\frac{\rho_0}{\rho_1(1-M_r)^2},$$

$$\mathrm{D}_{33} = \left\{\tfrac{1}{2}\alpha^2+\frac{\rho_1(1-M_r)^2}{\rho_0+\rho_1(1-M_r)^2}\frac{(x_3^2-x_2^2)}{|\boldsymbol{x}|^2}\right\}\frac{\rho_0}{\rho_1(1-M_r)^2},$$

$$\mathrm{D}_{ij} = \frac{2\rho_0x_ix_j}{\{\rho_0+\rho_1(1-M_r)^2\}\,|\boldsymbol{x}|^2}\quad\text{for}\quad i\neq j.$$

The ω-integral in (4.21) may be evaluated to give

$$\frac{\partial^2 G_1^c}{\partial y_i\,\partial y_j}(\boldsymbol{y},\tau) = \frac{\mathrm{D}_{ij}}{4\pi|\boldsymbol{x}|c_0^2}\frac{\rho_1(1-M_r)^2}{\rho_0}\frac{\partial^2}{\partial\tau^2}\delta(\tau-t^*). \tag{4.23}$$

We can express these time derivatives at constant $\boldsymbol{y}$ in terms of a Lagrangian time derivative at constant $\boldsymbol{\eta}$ since

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}\tau}\delta(\tau-t^*) &= \left(\frac{\partial}{\partial\tau}+v_i\frac{\partial}{\partial y_i}\right)\delta(\tau-t+(|\boldsymbol{x}|-y_1\cos\theta)/c_0)\\
&= (1-N_r)\frac{\partial}{\partial\tau}\delta(\tau-t^*),
\end{aligned} \tag{4.24}$$

where $D/D\tau$ is the material time derivative, $\boldsymbol{N}$ is the 'Mach number' based on the local fluid velocity and the exterior speed of sound $\boldsymbol{N} = \boldsymbol{v}/c_0$, and N_r is $v_1 x_1/(c_0|\boldsymbol{x}|)$. From (4.23) and (4.24)

$$\frac{\partial^2 G_1^c}{\partial y_i\,\partial y_j}(\boldsymbol{y},\tau) = \frac{\rho_1(1-M_r)^2}{\rho_0}\frac{D_{ij}}{4\pi|\boldsymbol{x}|c_0^2(1-N_r)}\frac{D}{D\tau}\left\{\frac{1}{1-N_r}\frac{D}{D\tau}\delta(\tau-t^*)\right\} \tag{4.25}$$

and (4.14) becomes

$$Hp'(\boldsymbol{x},t) = \frac{D_{ij}}{4\pi|\boldsymbol{x}|c_0^4}\int d\tau\left\{\int_{\nu_1}\frac{\rho^* T'_{ij}}{\rho(1-N_r)}\frac{D}{D\tau}\left(\frac{1}{(1-N_r)}\frac{D\delta}{D\tau}\right)d^3\boldsymbol{\eta} - \int_{\nu_1^{(0)}}(\rho_0 U_1^2\delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})\frac{\partial^2\delta}{\partial\tau^2}d^3\boldsymbol{y}\right\}. \tag{4.26}$$

The region ν_1 is a constant function of $\boldsymbol{\eta}$ so we may integrate by parts to obtain

$$Hp'(\boldsymbol{x},t) = \frac{D_{ij}}{4\pi|\boldsymbol{x}|c_0^4}\iint_{\nu_1}\delta(\tau-t^*)\frac{D}{D\tau}\left\{\frac{1}{1-N_r}\frac{D}{D\tau}\left(\frac{\rho^* T'_{ij}}{\rho(1-N_r)}\right)\right\}d^3\boldsymbol{\eta}\,d\tau. \tag{4.27}$$

The τ-integral can now be evaluated

$$Hp'(\boldsymbol{x},t) = \frac{D_{ij}T_{ij}}{4\pi|\boldsymbol{x}|c_0^4}, \tag{4.28}$$

where
$$T_{ij} = \int_{\nu_1}\left[\frac{D}{D\tau}\left\{\frac{1}{1-N_r}\frac{D}{D\tau}\left(\frac{\rho^* T'_{ij}}{\rho(1-N_r)}\right)\right\}\frac{d^3\boldsymbol{\eta}}{|1-N_r|}\right].$$

The square brackets denote that the function they enclose is to be evaluated at a retarded time τ^* satisfying

$$\tau^* = t - \{|\boldsymbol{x}| - \cos\theta y_1(\boldsymbol{\eta},\tau^*)\}/c_0, \tag{4.29}$$

where $\boldsymbol{y}(\boldsymbol{\eta},\tau^*) = \boldsymbol{\eta} + \int^{\tau^*}\boldsymbol{v}(\boldsymbol{\eta},\tau)\,d\tau$ and the lower limit in this integral is the reference time at which the Lagrangian and Eulerian axes coincide. We differentiate (4.29) to show

$$\frac{\partial\tau^*}{\partial\eta_i} = \frac{x_1}{|\boldsymbol{x}|c_0}\left\{\delta_{i1} + v_1\frac{\partial\tau^*}{\partial\eta_i}\right\}$$

or
$$\frac{\partial\tau^*}{\partial\eta_i} = \frac{1}{1-N_r}\frac{x_1\delta_{i1}}{|\boldsymbol{x}|c_0}.$$

Hence away from the flow Mach angle the variation in retarded time across the jet may be neglected and (4.28) is a convenient description of the noise produced by turbulence in the vicinity of the jet. T_{ij} is the source term found by Ffowcs Williams (1974) to be relevant in shear layer problems and here it is multiplied by a directional factor D_{ij}, which describes the transmission properties of a compact circular cylindrical jet. $D_{11} = \rho_0 x_1^2/\rho_1(1-M_r)^2|\boldsymbol{x}|^2$ and Mani's result that the interaction of a longitudinal quadrupole with a jet flow results in a Doppler amplification of the type $(1-M_r)^{-2}(1-N_r)^{-3}$ is immediate, and even higher powers of the Doppler factor accompany unsteady convection. We can use (4.28) to determine the way in which the sound field scales with the various jet parameters.

From the $\rho v'_i v'_j$ term in T'_{ij}, we find

$$\overline{p'(\boldsymbol{x},t)\,p'(\boldsymbol{x},t+\tau)} \sim \rho^2 M^8 a^2/|\boldsymbol{x}|^2, \tag{4.30}$$

where the bar denotes the time average. This is just Lighthill's scaling law.

The term in T'_{ij} due to the density difference $c_1^2(\rho-\rho_1)$ must be treated more carefully, because although $\Delta\rho = \rho-\rho_1$ may be large near the edge of the jet, $\mathrm{D}\rho/\mathrm{D}\tau$ is always small, and cannot be scaled as $\Delta\rho U_1/a$. Instead we expand

$$c_1^2\frac{\mathrm{D}}{\mathrm{D}\tau}\left(\frac{\rho^*(\rho-\rho_1)}{\rho(1-N_r)}\right) \quad \text{as} \quad \frac{c_1^2\rho^*\Delta\rho}{\rho(1-N_r)^2}\frac{\mathrm{D}N_r}{\mathrm{D}\tau}-\frac{c_1^2\rho^*}{(1-N_r)}\frac{\mathrm{D}}{\mathrm{D}\tau}\left(\frac{\rho_1}{\rho}\right).$$

The first term is the largest for small Mach numbers and we obtain

$$\overline{\rho'(\boldsymbol{x},t)\,\rho'(\boldsymbol{x},t+\tau)} \sim (\Delta\rho)^2 M^6a^2/|\boldsymbol{x}|^2. \tag{4.31}$$

Alternatively we could do the scaling in the frequency space and determine the spectral characteristics of the sound. The power spectral density $W(\boldsymbol{x},s)$ is defined by

$$W(\boldsymbol{x},s) = \int\overline{\rho'(\boldsymbol{x},t)\,\rho'(\boldsymbol{x},t+\tau)}\,\mathrm{e}^{-\mathrm{i}sU_1\tau/a}\,(U_1/a)\,\mathrm{d}\tau \tag{4.32}$$

or equivalently
$$W(\boldsymbol{x},s)\,\delta(\omega+\omega') = \overline{\rho'(\boldsymbol{x},\omega)\,\rho'(\boldsymbol{x},\omega')}\,U_1/2\pi a, \tag{4.33}$$

where $s = \omega a/U_1$ is the Strouhal number. The total acoustic power is then given by $\int W(\boldsymbol{x},s)\,\mathrm{d}s$.

When (4.22) is substituted into (4.16) we find

$$H\rho'(\boldsymbol{x},t) = -\frac{D_{ij}}{8\pi^2|\boldsymbol{x}|c_0^4}\int\omega^2\tilde{T}_{ij}(\boldsymbol{y},-\omega)\exp-\mathrm{i}\omega\{t-(|\boldsymbol{x}|-y_1\cos\theta)/c_0\}\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\omega \tag{4.34}$$

so that
$$H\rho'(\boldsymbol{x},\omega) = -\frac{\omega^2D_{ij}}{4\pi|\boldsymbol{x}|c_0^4}\int T'_{ij}(\boldsymbol{y},\omega)\exp-\mathrm{i}\omega(|\boldsymbol{x}|-y_1\cos\theta)/c_0\,\mathrm{d}^3\boldsymbol{y},$$

$T'_{ij}(\boldsymbol{y},\omega)$ is the Fourier transform of $\bar{H}T'_{ij}(\boldsymbol{y},\tau)$; or equivalently

$$H\rho'(\boldsymbol{x},\omega) = -\frac{\omega^2D_{ij}}{4\pi|\boldsymbol{x}|c_0^4}T'_{ij}(\boldsymbol{k}^*,\omega)\exp-\mathrm{i}\omega|\boldsymbol{x}|/c_0, \tag{4.35}$$

where $\boldsymbol{k}^* = (-\omega\cos\theta/c_0, 0, 0)$ and $T'_{ij}(\boldsymbol{k}^*,\omega)$ is the four-dimensional Fourier transform of $\bar{H}T'_{ij}$.

From (4.33) and (4.35) we see that the $\rho v'_iv'_j$ term in T'_{ij} gives $W(\boldsymbol{x},s) \sim \rho^2M^8s^3a^2/|\boldsymbol{x}|^2$. However if ρ_0 and ρ_1 are not equal, the largest contribution to T'_{ij} at low Mach number comes from the density term, $c_1^2(\rho-\rho_1)\,\delta_{ij}$. For low Mach number flows

$$\frac{\partial\rho}{\partial\tau} \approx -v_i\frac{\partial\rho}{\partial y_i} \approx -\frac{\partial}{\partial y_i}(v_i\Delta\rho),$$

so that $\omega\rho(\boldsymbol{y},\omega) \approx (\omega/c_0)\,(v_i\Delta\rho)\,(\boldsymbol{y},\omega)$ and

$$W(\boldsymbol{x},s) \sim (\Delta\rho)^2M^6s^3a^2/|\boldsymbol{x}|^2. \tag{4.36}$$

Near the jet axis or the mean flow Mach angle, or in the case of a very light jet, in fact whenever $\rho_1(1-M_r)^2 \leqslant \rho_0(\gamma a)^2\ln(\gamma_0a)$, this scaling is not valid because the second term in the denominator of $a_0(\omega)$ must be retained.

When the observer at $\boldsymbol{x}$ is near the axis of a jet of arbitrary density, $\sin\theta = 0 = \gamma_0$, then $a_n(\omega) = 0$ for all n, and G_1^c and all its derivatives are zero; no sound is heard.

In the case of a very light jet the compact limit of the Green function has a different form, and different scaling laws are obtained. For a jet which is lighter than it is compact (by that we mean $\rho_1 \ll \rho_0\epsilon^2\ln\epsilon$)

$$a_{-n}(\omega) = a_n(\omega) = \frac{\rho_1(1-M_r)^2}{2\pi\rho_0}\left(\frac{\gamma_0}{\gamma}\right)^n\{1+O(\epsilon^{2n}\ln\epsilon)+O(\epsilon^n)\} \quad n \geqslant 1$$

and $$a_0(\omega) = \frac{\rho_1(1-M_r)^2}{2\pi\rho_0(\gamma a)^2 \ln(|\omega|a/c_0)}\left\{1+O(\ln\epsilon)^{-1}+O\left(\frac{\rho_1(1-M_r)^2}{\rho_0\epsilon^2\ln\epsilon}\right)\right\},$$

for $\boldsymbol{x}$ such that $\alpha^2 = (1-M_r)^2 c_0^2/c_1^2 - \cos^2\theta \neq 0$. Then from (4.20)

$$\frac{\partial^2 G_1(\boldsymbol{y},\omega)}{\partial y_i\,\partial y_j} = -\frac{\rho_1(1-M_r)^2}{\rho_0 4\pi|\boldsymbol{x}|a^2}\frac{E_{ij}\,\mathrm{e}^{-\mathrm{i}\omega t^*}}{\ln(|\omega|a/c_0)}, \tag{4.37}$$

where $E_{11} = 2\cos^2\theta/\alpha^2$, $E_{22} = E_{33} = 1$, $E_{ij} = 0$ for $i \neq j$, and substitution into equation (4.16) gives

$$H\rho'(\boldsymbol{x},t) = -\frac{E_{ij}}{8\pi^2|\boldsymbol{x}|a^2c_0^2}\int\frac{T'_{ij}(\boldsymbol{y},-\omega)}{\ln(|\omega|a/c_0)}\exp\left[-\mathrm{i}\omega\{t-(|\boldsymbol{x}|-y_1\cos\theta)/c_0\}\right]\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\omega, \tag{4.38}$$

$$H\rho'(\boldsymbol{x},\omega) = -\frac{E_{ij}}{4\pi|\boldsymbol{x}|a^2c_0^2}\frac{T'_{ij}(\boldsymbol{k}^*,\omega)}{\ln(|\omega|a/c_0)}\exp(-\mathrm{i}\omega|\boldsymbol{x}|/c_0). \tag{4.39}$$

We can now use this expression for $\rho'(\boldsymbol{x},\omega)$ to determine how $W(\boldsymbol{x},s)$ depends on the various parameters. The $\rho v'_i v'_j$ term in T'_{ij} is largest near the surface where $\rho \approx \rho_0$ and

$$W(\boldsymbol{x},s) \sim \frac{\rho_0^2 M^4 a^2}{s\ln^2(sM)\,|\boldsymbol{x}|^2} \approx \rho_0^2 M^4 a^2/(s|\boldsymbol{x}|^2),$$

because for small M, the variation of $\ln(sM)$ with M is smaller than algebraic.

However, at low Mach numbers the main contribution to the sound field comes from the density term in T'_{ij}, and scaling that term gives

$$W(\boldsymbol{x},s) \sim \frac{(\Delta\rho)^2 M^2 a^2}{s\ln^2(sM)\,|\boldsymbol{x}|^2} \approx \frac{(\Delta\rho)^2M^2a^2}{s|\boldsymbol{x}|^2}, \quad \text{for} \quad \rho_1 \ll \rho_0 M^2 s^2 \ln(sM) \quad \text{and} \quad Ms \ll 1.$$

We see that in general the sound produced by turbulence within a very light jet scales with the second power of the Mach number, a most unexpected result which may well be relevant to the 'excess noise' problem. This scaling law predicts that the sound intensity is a factor $(Ms)^{-4}$ larger than the usual scaling law $(\Delta\rho)^2 M^6 s^3$.

For a point $\boldsymbol{x}$ on the edge of the 'zone of relative silence' where α is zero, i.e.

$$(1-M_r)^2 c_0^2 = c_1^2\cos^2\theta,$$

then even for a light jet the expression (4.22) for the derivatives of G_1^c holds and there must be a local peak in the far field intensity of noise from a light jet flow.

5. Conclusions

We have shown that the jet noise problem may be modelled by particle attached quadrupoles convected with the velocity of the actual fluid but positioned near a hypothetical instability free vortex sheet. The strength of each quadrupole is Lighthill's stress tensor per unit *mass*. Equation (2.30) expresses our main results. Mani's work has shown that this type of model agrees well with experiment and our theory justifies Mani's general procedure while establishing the source characteristics needed for an exact result. The exact sources differ from and are simpler than those used by Mani which contain spurious monopole and dipoles due to density gradient terms. The effect of density variation on the source is shown to vanish unless the density, or the velocity, of a moving material particle changes.

The deliberate emphasis of 'finiteness' rather than 'causality' in our analogy results in the shear layer's instability waves, as they grow into turbulence, being heard as sound that builds up as a precursor of the main turbulence driven field. Some jet flows will also support 'neutral' waves that travel without decay along the jet at constant speed. These might provide a means by which distant irrelevant events in the jet's history are retained as sound, but we show that to be impossible, only the *unsteady* part of the equivalent source field is non-zero. The sound in our exact analogy is totally uncoupled to any free waves of the basic flow.

We have examined the circular compact jet in some detail and, in addition to some previously known features of the mean flow/acoustic interaction, for example that no sound is heard on the jet axis (see Dash 1976), we have uncovered an interesting new aspect of the problem. Whenever the jet is very light it can provide a wave guide in which the effects of source activity persist for some time but eventually leak out as sound. This interaction drastically distorts the free field characteristics of the turbulent sources and in fact results in the Reynolds stress induced waves having an intensity that scales with only the fourth power of jet velocity rather than Lighthill's eighth power law for free quadrupoles. The source terms associated with density inhomogeneities have an even lower sensitivity to jet speed variation, the intensity of their sound scaling on the *square* of jet speed. We do not think this is a spurious artefact of the model and consider that it may have some bearing on the so-called 'excess noise' problem. There, the noise of a real hot jets is known to be much less sensitive to velocity changes than the U^8 dependence thought to be relevant to the 'pure mixing' noise of a low Mach number jet.

Our exact extension of the Lighthill theory includes mean flow acoustic interactions in a way that is analytically tractable, and which might be useful in describing the sound produced by real turbulent flows. We believe that in many applications our modelling will be as representative of the real thing as numerical calculations based on less tractable laminar flow simulations of the problem; the success of Mani's calculations supports this belief. But we do not claim, nor indeed do we think it to be true, that this vortex sheet model is any more correct than Lilley's. On the contrary, the main criticisms of that model rest on the inadequate treatment of the instability modes and the interpretation of the right hand side of the Lilley equation as the *source*. In facing that issue for the vortex sheet modelling, we have in effect also justified the procedure advocated by Lilley, and left the way open for other analogies based on different 'mean' velocity profiles. The relative merits of the various analogies now rest on their ease of handling, and we suspect that our analogy will be difficult to improve upon in that regard. We certainly do not believe this analogy to be in any sense unique, and can foresee that equally, and possibly more, penetrating analogies can be built upon linear models of thickening shear layers where the instability waves can remain bounded without the need for 'precursors'. Also nonlinear analogies are obviously possible in which the same boundedness can be ensured by finite amplitude effects. But both of these schemes are likely to be extremely difficult to deduce and eventually handle. It will not be easy to develop them to a stage where they are as powerful as the exact linear laminar parallel flow analogies of which ours is one member. These will have to form the basis of aerodynamic noise theory for some time yet.

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Appendix A

We now derive a representation for the sound field when the point source excites a neutral stability mode of the vortex sheet model and $\overline{H}G^{\dagger}$ is only bounded as $|\boldsymbol{y}|,\ -\tau\to\infty$.

We define a Heaviside function H, such that H is unity for points within both ν_0 and a sphere of large radius L, and zero elsewhere. $\partial H/\partial y_i$ is therefore non-zero on $S(\tau)$, the surface separating ν_0 from ν_1, and also on the surface of the sphere of radius L contained in ν_0. Equation (2.15) is still valid but now describes the flow within a finite region. We multiply it by $H^T(\tau)$, where H^T is unity for $|\tau| < T$, zero elsewhere, and rearrange to give

$$\left(\frac{\partial^2}{\partial\tau^2}-c_0^2\frac{\partial^2}{\partial y_i^2}\right)(H^TH\rho') = \frac{\partial^2(H^THT_{ij})}{\partial y_i\,\partial y_j}-\frac{\partial}{\partial y_j}\left(H^Tp_{ij}\frac{\partial H}{\partial y_i}\right)+\rho_0\frac{\partial}{\partial\tau}\left(H^Tv_i\frac{\partial H}{\partial y_i}\right)$$
$$+\frac{\partial}{\partial\tau}\left(H\rho'\frac{\partial H^T}{\partial\tau}\right)+\frac{\partial H^T}{\partial\tau}\frac{\partial(H\rho)}{\partial\tau}. \qquad \text{(A 1)}$$

It follows from the definition of $G_0^{\dagger}$ given in (2.11) that

$$H^TH\rho'(\boldsymbol{x},t) = -\int_{\infty} H^TH\rho'(\boldsymbol{y},\tau)\left\{\frac{\partial^2G_0^{\dagger}}{\partial y_i^2}-\frac{1}{c_0^2}\frac{\partial^2G_0^{\dagger}}{\partial\tau^2}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 2)}$$

We integrate this by parts exactly as before, the end-point contributions vanish because $H^T H\rho'(\boldsymbol{y},\tau)$ is zero as $|\boldsymbol{y},\tau|\to\infty$.

$$H^T H\rho'(\boldsymbol{x},t) = \int_\infty \frac{G_0^\dagger}{c_0^2}\left\{\frac{\partial^2}{\partial\tau^2}-c_0^2\frac{\partial^2}{\partial y_i^2}\right\} H^T H\rho'(\boldsymbol{y},\tau)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 3)}$$

We then substitute equation (A 1) into (A 3) and rearrange by partial integration to obtain

$$c_0^2 H^T H\rho'(\boldsymbol{x},t) = \int_\infty \left\{H^T H T_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i\,\partial y_j} + H^T p_{ij}\frac{\partial H}{\partial y_i}\frac{\partial G_0^\dagger}{\partial y_j} - H^T\rho_0\frac{\partial G_0^\dagger}{\partial\tau}v_i\frac{\partial H}{\partial y_i}\right.$$
$$\left. + \frac{\partial H^T}{\partial\tau}\left(\frac{\partial(H\rho)}{\partial\tau}G_0^\dagger - H\rho'\frac{\partial G_0^\dagger}{\partial\tau}\right)\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 4)}$$

A function Γ is introduced which satisfies $\partial\Gamma/\partial\tau = \partial G_0^\dagger/\partial\tau$ on those parts on $S(\tau)$ where $v_n \neq 0$, then equation (A 4) may be rewritten as

$$c_0^2 H^T H\rho'(\boldsymbol{x},t) = \int_\infty H^T\left\{H T_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i\,\partial y_j} + p_{ij}\frac{\partial H}{\partial y_i}\frac{\partial G_0^\dagger}{\partial y_j} + \bar{H}\rho_0\frac{\partial^2\Gamma}{\partial\tau^2}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau$$
$$+\int_\infty \frac{\partial H^T}{\partial\tau}\left\{\rho_0\bar{H}\frac{\partial\Gamma}{\partial\tau} - H\rho'\frac{\partial G_0^\dagger}{\partial\tau} + G_0^\dagger\frac{\partial(H\rho)}{\partial\tau}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 5)}$$

The last term of this equation can be simplified, because from the definition of H^T,

$$\int_{-\infty}^{\infty}(\partial H^T/\partial\tau)\,K(\boldsymbol{y},\tau)\,\mathrm{d}\tau = -\{K(\boldsymbol{y},T) - K(\boldsymbol{y},-T)\}$$

for any function K. Far from the sources in the real flow the fluid is undisturbed and therefore the perturbation flow parameters $\boldsymbol{v}, p, \rho'$ and their derivatives vanish on the surface of the sphere of radius L or at $\tau = -T$ for all $\boldsymbol{y}$, and $G^\dagger = 0$ at $\tau = T$ provided only that L and T are large enough. Equation (A 5) then simplifies to

$$c_0^2 H^T H\rho'(\boldsymbol{x},t) = \int_{-T}^{T}\int\left\{H T_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i\,\partial y_j} + p_{ij}\frac{\partial H}{\partial y_i}\frac{\partial G_0^\dagger}{\partial y_j} + \bar{H}\rho_0\frac{\partial^2\Gamma}{\partial\tau^2}\right\}\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau + \left[\int\rho_0\bar{H}\frac{\partial\Gamma}{\partial\tau}\mathrm{d}^3\boldsymbol{y}\right]_{\tau=-T}. \qquad \text{(A 6)}$$

The region in the interior of the jet may be treated in exactly the same way. $\bar{H}$ is defined to be unity for points within both ν_1 and the sphere of radius L, and zero elsewhere. We then multiply (2.18) by H^T and rearrange

$$\left\{\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}-c_1^2\frac{\partial^2}{\partial y_i^2}\right\}\{H^T(\bar{H}-H_V)\,\rho_1'\} = \frac{\partial^2}{\partial y_i\,\partial y_j}\{H^T(\bar{H}-H_V)\,T'_{ij}\} - \frac{\partial}{\partial y_j}\left\{H^T p_{ij}\left(\frac{\partial\bar{H}}{\partial y_i}-\frac{\partial H_V}{\partial y_i}\right)\right\}$$
$$-\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}(H^T\bar{H}) + \rho_1\frac{\mathrm{D}_1}{\mathrm{D}\tau}\left(H^T\frac{\mathrm{D}_1 H_V}{\mathrm{D}\tau}\right) + \frac{\mathrm{D}_1 H^T}{\mathrm{D}\tau}\frac{\mathrm{D}_1}{\mathrm{D}\tau}\{(\bar{H}-H_V)\,\rho\}$$
$$+\frac{\mathrm{D}_1}{\mathrm{D}\tau}\left\{\frac{\mathrm{D}_1 H^T}{\mathrm{D}\tau}(\rho(\bar{H}-H_V)+\rho_1 H_V)\right\}. \qquad \text{(A 7)}$$

The definition of $G_1^\dagger$ (2.12) gives

$$0 = \int_\infty H^T(\bar{H}-H_V)\,\rho_1'(\boldsymbol{y},\tau)\left\{\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}-c_1^2\frac{\partial^2}{\partial y_i^2}\right\}G_1^\dagger\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 8)}$$

The differential operators can be transferred into $H^T(\bar{H}-H_V)\,\rho_1'$ by partial integration; contributions at infinity vanish because $H^T(\bar{H}-H_V)\,\rho_1'$ is zero as $|\boldsymbol{y},\tau|\to\infty$.

$$0 = \int_\infty G_1^\dagger\left\{\frac{\mathrm{D}_1^2}{\mathrm{D}\tau^2}-c_1^2\frac{\partial^2}{\partial y_i^2}\right\}H^T(\bar{H}-H_V)\,\rho_1'(\boldsymbol{y},\tau)\,\mathrm{d}^3\boldsymbol{y}\,\mathrm{d}\tau. \qquad \text{(A 9)}$$

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We substitute (A 7) into (A 9) and rearrange by partial integration, end-point contributions are again zero because $H^T(\bar{H}-H_V)\,\rho_1'$ is zero as any one of the variables tends to $\pm$ infinity.

$$0=\int_\infty H^T\left\{(\bar{H}-H_V)\,T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}+p_{ij}\frac{\partial G_1^\dagger}{\partial y_j}\left(\frac{\partial \bar{H}}{\partial y_i}-\frac{\partial H_V}{\partial y_i}\right)-\rho_1\bar{H}\frac{D_1^2 G_1^\dagger}{D\tau^2}-\rho_1\frac{D_1 G_1^\dagger}{D\tau}\frac{D_1 H_V}{D\tau}\right\}d^3\boldsymbol{y}\,d\tau$$
$$+\int_\infty\frac{D_1 H^T}{D\tau}\left\{G_1^\dagger\frac{D_1}{D\tau}((\bar{H}-H_V)\,\rho)-\frac{D_1 G_1^\dagger}{D\tau}\rho(\bar{H}-H_V)-\rho_1 H_V\frac{D_1 G_1^\dagger}{D\tau}\right\}d^3\boldsymbol{y}\,d\tau, \tag{A 10}$$

which for a large enough value of T may be written:

$$0=\int_{-T}^{T}\int_\infty\left\{(\bar{H}-H_V)\,T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}+p_{ij}\frac{\partial G_1^\dagger}{\partial y_j}\left(\frac{\partial \bar{H}}{\partial y_i}-\frac{\partial H_V}{\partial y_i}\right)-\rho_1\bar{H}\frac{D_1^2 G_1^\dagger}{D\tau^2}-\rho_1\frac{D_1 G_1^\dagger}{D\tau}\frac{D_1 H_V}{D\tau}\right\}d^3\boldsymbol{y}\,d\tau$$
$$+\left[\int\!\!\int\left\{-\rho_0\bar{H}\frac{D_1 G_1^\dagger}{D\tau}+(\rho_0-\rho_1)\,H_V\frac{D_1 G_1^\dagger}{D\tau}+\rho_0 G_1^\dagger U_1\left(\frac{\partial \bar{H}}{\partial y_1}-\frac{\partial H_V}{\partial y_1}\right)\right\}d^3\boldsymbol{y}\right]_{\tau=-T}. \tag{A 11}$$

This equation is multiplied by $\beta(\boldsymbol{x},t)$ and the result added to (A 6) to obtain the exact equation

$$c_0^2 H^T H\rho'(\boldsymbol{x},t)=\int_{-T}^{T}\int_\infty\left\{HT_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i\,\partial y_j}+\beta(\bar{H}-H_V)\,T'_{ij}\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j}\right.$$
$$\left.-\beta p_{ij}\frac{\partial H_V}{\partial y_i}\frac{\partial G_1^\dagger}{\partial y_j}-\beta\rho_1\frac{D_1 G_1^\dagger}{D\tau}\frac{D_1 H_V}{D\tau}\right\}d^3\boldsymbol{y}\,d\tau+\Phi+\Psi, \tag{A 12}$$

where
$$\Phi(\boldsymbol{x},t)=\int_{-T}^{T}\int_\infty\left\{p_{ij}\frac{\partial \bar{H}}{\partial y_i}\left(\beta\frac{\partial G_1^\dagger}{\partial y_j}-\frac{\partial G_0^\dagger}{\partial y_j}\right)+\bar{H}\left(\rho_0\frac{\partial^2\Gamma}{\partial\tau^2}-\beta\rho_1\frac{D_1^2 G_1^\dagger}{D\tau^2}\right)\right\}d^3\boldsymbol{y}\,d\tau$$
and
$$\Psi(\boldsymbol{x},t)=\left[\int_\infty\left\{-\beta\rho_0\bar{H}\frac{D_1 G_1^\dagger}{D\tau}+\beta(\rho_0-\rho_1)\,H_V\frac{D_1 G_1^\dagger}{D\tau}+\beta\rho_0 G_1^\dagger U_1\left(\frac{\partial \bar{H}}{\partial y_1}-\frac{\partial H_V}{\partial y_1}\right)+\rho_0\bar{H}\frac{\partial\Gamma}{\partial\tau}\right\}d^3\boldsymbol{y}\right]_{-T}.$$

This equation differs from that obtained previously (equation (2.22)) only in as much as the integration range is now finite. The additional term Ψ is due to any source terms at $\tau=-T$, the effect of which lives forever when the vortex sheet resonances are excited.

We now choose $G^\dagger$ to satisfy jump conditions across S that minimize Φ, and as in (2.25) we pick $G_0^\dagger$ and $G_1^\dagger$ such that

$$\frac{\partial G_0^\dagger}{\partial n}=\beta\frac{\partial G_1^\dagger}{\partial n}\quad\text{and}\quad\frac{\partial G_0^\dagger}{\partial\tau}=\frac{\partial\Gamma}{\partial\tau} \tag{A 13}$$

on all parts of $S(\tau)$ where $v_n\neq 0$, $p_{ij}\neq 0$. Γ is defined in ν_1 by $\bar{H}\rho_0\,\partial^2\Gamma/\partial\tau^2=\bar{H}\beta\rho_1 D_1^2 G_1^\dagger/D\tau^2$. With this choice of $G^\dagger$, Φ depends only on the viscous stress on the surface $S(\tau)$ (just as in (2.26)), but at a high Reynolds number the sound produced by this viscous term will be negligible compared with that due to the inertial terms, and so we neglect it.

The term Ψ is actually independent of the real flow and depends only on the initial conditions. We denote the initial position of the fluid surface $S(\tau)$ by S_0 and the region contained within S_0 by $\nu_1^{(0)}$. We then introduce $\bar{H}_0$, a Heaviside function which is unity for points within both $\nu_1^{(0)}$ and the large sphere of radius L, and zero elsewhere. Therefore

$$\bar{H}_0(\boldsymbol{y})=\lim_{\tau\to-\infty}\bar{H}(\boldsymbol{y},\tau).$$

Similarly the starting position of the surface $\Sigma(\tau)$ is denoted by Σ_0 and we introduce H_{V0} the Heaviside function which is unity within Σ_0 and zero elsewhere.

Then

$$\Psi = \left[\int_\infty \left\{-\beta\rho_0 \bar{H}_0 \frac{D_1 G_1^\dagger}{D\tau} + \beta(\rho_0-\rho_1) H_{V0}\frac{D_1 G_1^\dagger}{D\tau} + \beta\rho_0 G_1^\dagger U_1\left(\frac{\partial \bar{H}_0}{\partial y_1} - \frac{\partial H_{V0}}{\partial y_1}\right) + \rho_0 \bar{H}_0 \frac{\partial \Gamma}{\partial \tau}\right\} d^3\boldsymbol{y}\right]_{\tau=-T} \tag{A 14}$$

$$= \int_{-T}^{T}\int_\infty \frac{\partial}{\partial \tau}\left\{\beta\rho_0 \bar{H}_0 \frac{D_1 G_1^\dagger}{D\tau} - \beta(\rho_0-\rho_1) H_{V0}\frac{D_1 G_1^\dagger}{D\tau} - \beta\rho_0 G_1^\dagger U_1\left(\frac{\partial \bar{H}_0}{\partial y_1} - \frac{\partial H_{V0}}{\partial y_1}\right) - \rho_0 \bar{H}_0 \frac{\partial \Gamma}{\partial \tau}\right\} d^3\boldsymbol{y}\, d\tau. \tag{A 15}$$

The right hand side may be rearranged to give

$$\Psi = -\beta \int_{-T}^{T}\int_\infty \left\{(\bar{H}_0 - H_{V0})\,(\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})\,\frac{\partial^2 G_1^\dagger}{\partial y_i\,\partial y_j} - \rho_1 \frac{D_1 G_1^\dagger}{D\tau} U_1 \frac{\partial H_{V0}}{\partial y_1}\right\} d^3\boldsymbol{y}\, d\tau. \tag{A 16}$$

From (A 12) we finally obtain our modified vortex sheet analogy: $c_0^2 H^T H\rho'(\boldsymbol{x}, t)$

$$= \int_{-T}^{T}\int_\infty \left\{HT_{ij}\frac{\partial^2 G_0^\dagger}{\partial y_i \partial y_j} + \beta\Big((\bar{H} - H_V)\,T'_{ij} - (\bar{H}_0 - H_{V0})\,(\rho_0 U_1^2\delta_{i1}\delta_{j1} - c_1^2(\rho_0-\rho_1)\,\delta_{ij})\Big)\frac{\partial^2 G_1^\dagger}{\partial y_i \partial y_j}\right.$$
$$\left. - \beta p_{ij}\frac{\partial H_V}{\partial y_i}\frac{\partial G_1^\dagger}{\partial y_j} - \beta\rho_1 \frac{D_1 G_1^\dagger}{D\tau}\left(\frac{D_1 H_V}{D\tau} - U_1 \frac{\partial H_{V0}}{\partial y_1}\right) + e_{ij}\frac{\partial \bar{H}}{\partial y_i}\frac{\partial}{\partial y_j}(G_0^\dagger - \beta G_1^\dagger)\right\} d^3\boldsymbol{y}\, d\tau. \tag{A 17}$$

The terms in (A 17) arising from Ψ conveniently ensure that the source terms decay as any one of the variables tends to infinity because

$$(\bar{H} - H_V)\,T'_{ij} \to (\bar{H}_0 - H_{V0})\,(\rho_0 U_1^2 \delta_{i1}\delta_{j1} - c_1^2(\rho_0 - \rho_1)\,\delta_{ij})$$

and

$$\frac{D_1 H_V}{D\tau} \to U_1 \frac{\partial H_{V0}}{\partial y_1}, \quad \text{as} \quad |\boldsymbol{y}|,\ -\tau \to \infty.$$

When $G^\dagger$ decays at infinity, $\Psi = 0$. Then (A. 17) and the previous representation, (2.26) are entirely equivalent; either form may be used.

Appendix B. The reciprocal theorem

When the surface is only linearly disturbed as described in §3, we can determine a simple relation between the reciprocal Green function defined by (2.11) and (2.12) with the jump conditions (3.5), and the Green function defined in (3.6), when $\bar{H}G^\dagger \to 0$ as $|\boldsymbol{y}, \tau| \to \infty$.

Throughout this section $\boldsymbol{y}$ is a point within the jet, i.e. in the region $\nu_1^{(0)}$, and $\boldsymbol{x}$ is in the ambient fluid $\nu_1^{(0)}$. $H_0(\boldsymbol{z})$ is a Heaviside function such that

$$H_0 = 1 \quad \text{for} \quad \boldsymbol{z} \quad \text{in} \quad \nu_0^{(0)}$$
$$= 0 \quad \text{for} \quad \boldsymbol{z} \quad \text{in} \quad \nu_1^{(0)}, \tag{B 1}$$

and $\bar{H}_0 = 1 - H_0$.

$H_0(\boldsymbol{z})$ is independent of z_1 because S_0, the bounding surface between $\nu_0^{(0)}$ and $\nu_1^{(0)}$, is a fixed surface parallel to the z_1-axis.

Now
$$H_0(\boldsymbol{x})\, G(\boldsymbol{x}, t|\boldsymbol{y}, \tau) = \int H_0(\boldsymbol{z})\, G(\boldsymbol{z}, s|\boldsymbol{y}, \tau)\, \delta(\boldsymbol{x}-\boldsymbol{z}, t-s)\, d^3\boldsymbol{z}\, ds \tag{B 2}$$

$$= -\int H_0(\boldsymbol{z})\, G(\boldsymbol{z}, s|\boldsymbol{y}, \tau)\left(\frac{\partial^2}{\partial z_i^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial s^2}\right) G_0^\dagger(\boldsymbol{z}, s|\boldsymbol{x}, t)\, d^3\boldsymbol{z}\, ds \tag{B 3}$$

from (2.11). After integration by parts and use of equation (3.6), we obtain

$$H_0(\boldsymbol{x})\,G(\boldsymbol{x},t|\boldsymbol{y},\tau) = \int \frac{\partial H_0}{\partial z_i}\left\{G(\boldsymbol{z},s|\boldsymbol{y},\tau)\frac{\partial G_0^\dagger}{\partial z_i}(\boldsymbol{z},s|\boldsymbol{x},t) - G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\frac{\partial G}{\partial z_i}(\boldsymbol{z},s|\boldsymbol{y},\tau)\right\}\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s; \tag{B 4}$$

contributions at infinity vanish because G represents an outgoing, and $G^\dagger$ an incoming wave in the variables $(\boldsymbol{z}, s)$. $H_0(\boldsymbol{x})$ is independent of x_1 and t, hence

$$\begin{aligned} H_0(\boldsymbol{x})\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}G(\boldsymbol{x},t|\boldsymbol{y},\tau) &= \frac{\mathrm{D}_1^2}{\mathrm{D}t^2}\{H_0(\boldsymbol{x})\,G(\boldsymbol{x},t|\boldsymbol{y},\tau)\} \\ &= \int \frac{\partial H_0}{\partial z_i}\left\{G(\boldsymbol{z},s|\boldsymbol{y},\tau)\frac{\partial}{\partial z_i}\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\right. \\ &\qquad \left. -\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\frac{\partial G}{\partial z_i}(\boldsymbol{z},s|\boldsymbol{y},\tau)\right\}\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s \end{aligned} \tag{B 5}$$

by differentiating (B. 4). By applying (3.2) we can write

$$\begin{aligned} H_0(\boldsymbol{x})\frac{\mathrm{D}_1^2}{\mathrm{D}t^2}G(\boldsymbol{x},t|\boldsymbol{y},\tau) &= \int \frac{\partial H_0}{\partial z_i}\left\{G(\boldsymbol{z},s|\boldsymbol{y},\tau)\frac{\partial \mathrm{D}_1^2}{\partial z_i\,\mathrm{D}s^2}G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\right. \\ &\qquad \left. -\frac{\mathrm{D}_1^2}{\mathrm{D}s^2}G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\frac{\partial}{\partial z_i}G(\boldsymbol{z},s|\boldsymbol{y},\tau)\right\}\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s. \end{aligned} \tag{B 6}$$

We impose the condition

$$H_0(\boldsymbol{z})\,G(\boldsymbol{z},s|\boldsymbol{y},\tau)\to 0 \quad \text{as} \quad |z_1|,\,|s|\to\infty \tag{B 7}$$

then an integration by parts of the last term in (B 6) gives

$$\begin{aligned} \rho_1 H_0(\boldsymbol{x})\frac{\mathrm{D}_1^2 G}{\mathrm{D}t^2}(\boldsymbol{x},t|\boldsymbol{y},\tau) &= \iint\left\{G(\boldsymbol{z},s|\boldsymbol{y},\tau)\,\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}s^2}\frac{\partial G_0^\dagger}{\partial n}(\boldsymbol{z},s|\boldsymbol{x},t)\right. \\ &\qquad \left. -G_0^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\,\rho_1\frac{\mathrm{D}_1^2}{\mathrm{D}s^2}\frac{\partial G}{\partial n}(\boldsymbol{z},s|\boldsymbol{y},\tau)\right\}\mathrm{d}S_0\,\mathrm{d}s. \end{aligned} \tag{B 8}$$

Similarly

$$\bar{H}_0(\boldsymbol{y})G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = \int \bar{H}_0(\boldsymbol{z})\,G_1^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\,\delta(\boldsymbol{y}-\boldsymbol{z},\tau-s)\,\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s \tag{B 9}$$

$$= -\int \bar{H}_0(\boldsymbol{z})\,G_1^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\left\{\frac{\partial^2}{\partial z_i^2}-\frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}s^2}\right\}G(\boldsymbol{z},s|\boldsymbol{y},\tau)\,\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s \tag{B 10}$$

from (3.6).

After integration by parts this becomes

$$\bar{H}_0(\boldsymbol{y})\,G_1^\dagger(\boldsymbol{y},\tau|\boldsymbol{x},t) = \int \frac{\partial \bar{H}_0}{\partial z_i}\left\{G_1^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\frac{\partial G}{\partial z_i}(\boldsymbol{z},s|\boldsymbol{y},\tau) - G(\boldsymbol{z},s|\boldsymbol{y},\tau)\frac{\partial G_1^\dagger}{\partial z_i}(\boldsymbol{z},s|\boldsymbol{x},t)\right\}\mathrm{d}^3\boldsymbol{z}\,\mathrm{d}s \tag{B 11}$$

because from (2.12),

$$\bar{H}_0(\boldsymbol{z})\left\{\frac{\partial^2}{\partial z_i^2}-\frac{1}{c_1^2}\frac{\mathrm{D}_1^2}{\mathrm{D}s^2}\right\}G_1^\dagger(\boldsymbol{z},s|\boldsymbol{x},t) = 0, \tag{B 12}$$

and the condition $\bar{H}G^\dagger\to 0$ as $|z_1|,\,|s|\to\infty$ ensures that the integrated terms vanish. It can easily be shown by differentiation of (B. 11) that

$$\begin{aligned} \bar{H}_0(\boldsymbol{y})\,\rho_0\frac{\partial^2 G_1^\dagger}{\partial t^2}(\boldsymbol{y},\tau|\boldsymbol{x},t) &= \iint\left\{G(\boldsymbol{z},s|\boldsymbol{y},\tau)\,\rho_0\frac{\partial^2}{\partial s^2}\frac{\partial G_1^\dagger}{\partial n}(\boldsymbol{z},s|\boldsymbol{x},t)\right. \\ &\qquad \left. -G_1^\dagger(\boldsymbol{z},s|\boldsymbol{x},t)\,\rho_0\frac{\partial^2}{\partial s^2}\frac{\partial G}{\partial n}(\boldsymbol{z},s|\boldsymbol{y},\tau)\right\}\mathrm{d}S_0\,\mathrm{d}s. \end{aligned} \tag{B 13}$$

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Both G and $G^{\dagger}$ satisfy the jump conditions (3.5) and hence we can deduce from (B 8) and (B 13)

$$\rho_0 \frac{\partial^2}{\partial t^2} G_1^{\dagger}(\boldsymbol{y}, \tau | \boldsymbol{x}, t) = \rho_1 \frac{\mathrm{D}_1^2 G}{\mathrm{D} t^2}(\boldsymbol{x}, t | \boldsymbol{y}, \tau), \tag{B 14}$$

which for $\boldsymbol{x}$ in the far field reduces to the required reciprocal identity

$$G(\boldsymbol{y}, -\tau | \boldsymbol{x}, -t) = G_1^{\dagger}(\boldsymbol{y}, \tau | \boldsymbol{x}, t) = \rho_1 \rho_0^{-1}(1 - M_r)^2 G(\boldsymbol{x}, t | \boldsymbol{y}, \tau). \tag{B 15}$$

We can now see that the two apparently different conditions we had to impose, (B 7) and that following equation (B 12), are in fact one and the same. The latter states that $G_1^{\dagger}(\boldsymbol{y}, \tau | \boldsymbol{x}, t) \to 0$ as $|y_1|, |\tau| \to \infty$ where $\boldsymbol{y}$ is in $\nu_1^{(0)}$, $\boldsymbol{x}$ is in $\nu_0^{(0)}$. But $G^{\dagger}$ is a function of y_1 and τ only in the combination $y_1 - x_1$, $\tau - t$ and so necessarily

$$G_1^{\dagger}(\boldsymbol{y}, \tau | \boldsymbol{x}, t) \to 0 \quad \text{as} \quad |x_1|, |t| \to \infty,$$

which from (B 15) gives condition (B 7),

$$G(\boldsymbol{x}, t | \boldsymbol{y}, \tau) \to 0 \quad \text{as} \quad |x_1|, |t| \to \infty.$$

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